

Stokes phenomena and non-perturbative completion in the multi-cut two-matrix models

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Abstract

The Stokes multipliers in the matrix models are invariants in the string-theory moduli space and related to the D-instanton chemical potentials. They not only represent non-perturbative information but also play an important role in connecting various perturbative string theories in the moduli space. They are a key concept to the non-perturbative completion of string theory and also expected to imply some remnant of strong coupling dynamics in M theory. In this paper, we investigate the non-perturbative completion problem consisting of two constraints on the Stokes multipliers. As the first constraint, Stokes phenomena which realize the multi-cut geometry are studied in the \mathbb{Z}_k symmetric critical points of the multi-cut two-matrix models. Sequence of solutions to the constraints are obtained in general k -cut critical points. A discrete set of solutions and a continuum set of solutions are explicitly shown, and they can be classified by several constrained configurations of the Young diagram. As the second constraint, we discuss non-perturbative stability of backgrounds in terms of the Riemann-Hilbert problem. In particular, our procedure in the 2-cut (1, 2) case (pure-supergravity case) completely fixes the D-instanton chemical potentials and results in the Hastings-McLeod solution to the Painlevé II equation. It is also stressed that the Riemann-Hilbert approach realizes an off-shell background independent formulation of non-critical string theory.

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1 Introduction

Non-critical string theory [1] has provided interesting theoretical laboratories which uncover various intriguing features about string theory. This string theory is known as solvable system not only in the perturbative world-sheet formulation, Liouville theory [2–10], but also in the non-perturbative matrix-model formulation [11–32]. Recently, among various kinds of matrix models, the multi-cut matrix models [33] have turned out to be a fruitful system. The first discovery was on the two-cut matrix models [34–39], which were found to describe type 0 superstring theory [40–42]. Furthermore, the multi-cut two-matrix models were generally found to have a correspondence with the so-called fractional superstring theory [43] and also with non-critical M theory as its strong-coupling dual theory [44], which realizes the philosophy proposed in the Hořava-Keeler non-critical M theory [45].

Quantitative analyses of critical points and perturbative amplitudes in the multi-cut two-matrix models have been carried out in [44, 46]. The main observables used there are macroscopic loop amplitudes (or resolvent) [12–15, 17, 18, 29, 30, 47–51] which provide the information of spectral curves, the classical spacetime of this string theory [32, 52, 53]. A concrete expression for spectral curve is important because it provides relevant information for reproducing all order perturbative amplitudes in the multi-cut two-matrix models by the method of topological recursions [54].

The main theme in this paper is, on the other hand, about *non-perturbative aspects of the multi-cut two-matrix models*. Non-perturbative aspects in matrix models have also been studied extensively [24, 27, 28, 31, 32, 51–53, 55–80].¹ The main concern is about non-perturbative contributions to the matrix-model free energy $\mathcal{F}(\mathcal{C}; g_{\text{str}})$ on the large N spectral curve \mathcal{C} :²

$$\mathcal{F}(\mathcal{C}; g_{\text{str}}) \underset{\text{asym}}{\simeq} \sum_{n=0}^{\infty} g_{\text{str}}^{2n-2} \mathcal{F}_n(\mathcal{C}) + \mathcal{F}_{\text{non-perturb.}}(\mathcal{C}; g_{\text{str}}), \quad g_{\text{str}} \rightarrow 0. \quad (1.1)$$

Here $\mathcal{F}_n(\mathcal{C})$ is the genus- n perturbative free energy on the spectral curve \mathcal{C} , and the information of the matrix-model potentials (so-called KP flows $\{t_n\}_{n \in \mathbb{Z}}$) is implicitly included in the spectral curve:

$$\mathcal{C} = \mathcal{C}(\{t_n\}_{n \in \mathbb{Z}}), \quad \{t_n\}_{n \in \mathbb{Z}} \in \mathcal{M}_{\text{string}}^{(\text{non-norm.})} \subset \mathbb{C}^{\infty}. \quad (1.2)$$

Here $\mathcal{M}_{\text{string}}^{(\text{non-norm.})}$ stands for the non-normalizable string-theory moduli space [81].³ The first quantitative implication was given in the early 90’s and is about the strength of string non-perturbative corrections which are of order $\mathcal{O}(e^{-1/g_{\text{str}}})$ quantities [82], i.e. open-string (D-brane) degree of freedom [83]:

$$\mathcal{F}_{\text{non-perturb.}}(\mathcal{C}; g_{\text{str}}) = \sum_I \theta_I \exp \left[-\frac{1}{g_{\text{str}}} \mathcal{S}_{\text{inst}}^{(I)}(\mathcal{C}; g_{\text{str}}) \right]. \quad (1.3)$$

¹See [80] for a nice review of these recent progress.

²We carefully put “asym” below the equation in order to emphasize that they are equal only in the asymptotic sense.

³The normalizable string-theory moduli space $\mathcal{M}_{\text{string}}^{(\text{norm.})}$ is known as the space of filling fraction [76] which parametrizes the on-shell string backgrounds. The off-shell backgrounds are defined in Section 5.

Here I is a set of indices which labels multi-instanton sectors, $I = \{i_1, i_2, \dots\}$,

$$\mathcal{S}_{\text{inst}}^{(I)}(\mathcal{C}; g_{\text{str}}) = \sum_{i \in I = \{i_1, i_2, \dots\}} \mathcal{S}_{\text{inst}}^{(i)}(\mathcal{C}) + \mathcal{O}(g_{\text{str}}). \quad (1.4)$$

Each primitive instanton action $\mathcal{S}_{\text{inst}}^{(i)}(\mathcal{C})$ ($i = 1, 2, \dots, N_{\text{inst}}$), is shown to correspond to a singular point of the spectral curve \mathcal{C} [42, 56–60, 62, 69] and is identified with the ZZ-brane disk amplitudes in Liouville theory [7, 9, 10]. It is worth mentioning that these instanton corrections including higher order g_{str} corrections $\mathcal{S}_{\text{inst}}^{(I)}(\mathcal{C}; g_{\text{str}})$ are generally expressed as theta functions on the spectral curve [55, 73] and important in order to make the free energy $\mathcal{F}(\mathcal{C}; g_{\text{str}})$ modular invariant under modular transformations of the spectral curve \mathcal{C} and also to be background independent in the normalizable string-theory moduli space $\mathcal{M}_{\text{string}}^{(\text{norm.})}$ (i.e. the filling fractions) [73, 76]. The constant θ_I is called D-instanton chemical potential (or fugacity). These constants are understood as integration constants of corresponding string equations [28], that is,

$$\frac{\partial \theta_I}{\partial t_m} = 0, \quad m \in \mathbb{Z}, \quad \{t_n\}_{n \in \mathbb{Z}} \in \mathcal{M}_{\text{string}}^{(\text{non-norm.})}, \quad (1.5)$$

for the flows in the non-normalizable moduli space $\mathcal{M}_{\text{string}}^{(\text{non-norm.})}$. It was shown [32] that the only N_{inst} (i.e. the number of primitive instantons) chemical potentials θ_i ($i = 1, 2, \dots, N_{\text{inst}}$) are independent among all the chemical potentials θ_I .

Although various aspects of matrix models have been understood well so far, there still remains an important issue regarding the D-instanton chemical potentials. This is also known as non-perturbative ambiguities of string theory. Therefore, *what is the physical requirement to determine the D-instanton chemical potentials?* Although the actual matrix models should employ some particular universal values [61], they seem to be totally free parameters at least within continuum formulations based on string (or loop) equations. This point has been studied in the bosonic minimal/2D string theories [61, 65, 66, 68], in the type 0 (1, 2) superstring theory [63], in the collective string field theory [64], in the free-fermion formulation [51, 67] and in the topological string interpretations [76]. In this paper, we address this issue by solving *non-perturbative completion problem* within a continuous formulation for the critical points of the multi-cut two-matrix models. In practice, we pick up physically acceptable D-instanton chemical potentials which realize physically reasonable behaviors in the non-perturbative regime $g_{\text{str}} \rightarrow \infty$. Our solutions are based on two physical requirements: One is *multi-cut boundary condition* (in Section 4) and the other is *non-perturbative stability of perturbative backgrounds* (in Section 5).

The first requirement, the multi-cut boundary condition, is a non-perturbative constraint on the Baker-Akhiezer function system in these multi-cut critical points:

$$g_{\text{str}} \frac{\partial}{\partial \zeta} \Psi(t; \zeta) = \mathcal{Q}(t; \zeta) \Psi(t; \zeta), \quad g_{\text{str}} \frac{\partial}{\partial t} \Psi(t; \zeta) = \mathcal{P}(t; \zeta) \Psi(t; \zeta), \quad (1.6)$$

where the equation system here is expressed as an ordinary differential equation in ζ and its isomonodromy deformation system in t .⁴ Note that the Lax operators in Eq. (1.6)

⁴ The parameter t is one of the parameters in the non-normalizable moduli space $\mathcal{M}_{\text{string}}^{(\text{non-norm.})}$, which is usually a coupling of the most relevant operator or the world-sheet cosmological constant.

in the k -cut critical points are $k \times k$ matrix-valued operators [69]. The idea of the first constraint is motivated by the non-perturbative relationship between the Baker-Akhiezer functions and cuts in the resolvent curves. This kind of relationship is discussed in terms of Airy function [52]. Specifically, the asymptotic expansion of the Airy function around the cut ($\zeta \rightarrow -\infty$) is expressed as⁵

$$\text{Ai}(t; \zeta) \underset{asym}{\simeq} \left(\frac{g_{\text{str}} \pi}{(\zeta + t)^{1/2}} \right)^{1/2} \left[e^{-\frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}} + i e^{\frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}} \right] + \dots, \quad (1.7)$$

where the relation to the resolvent (or macroscopic loop) operator $\mathcal{R}(\zeta)$ [20] is roughly expressed as

$$\text{Ai}(t; \zeta) \sim \exp \left[N \int^\zeta d\zeta' \mathcal{R}(\zeta') \right], \quad \mathcal{R}(\zeta) \equiv \frac{1}{N} \left\langle \text{tr} \frac{1}{\zeta - M} \right\rangle - \frac{V'(\zeta)}{2} \sim \sqrt{\zeta + t}, \quad (1.8)$$

with the expectation value $\langle \dots \rangle$ which is taken with respect to the Hermitian one-matrix model of a matrix M . From this expression, one observes that *the cut in the negative axes ($\zeta < -t$) appears as a line where a competition between the exponents $e^{\pm \frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}}$ (i.e. along the Stokes lines) happens*. Therefore, we interpret this as *a non-perturbative definition of the resolvent cuts*. This consideration turns out to be important in the fractional-superstring critical points of the multi-cut two-matrix models [44], since most of the cuts in these critical points are created by this procedure and cannot be read from the algebraic equations of the resolvent spectral curve. However, as we will see in Section 4, this procedure do not necessarily create the necessary and sufficient k cuts on the resolvent curve, even though the k -cut Baker-Akhiezer function Eq. (1.6) is obtained from the assumption that the critical points have k cuts around $\zeta \rightarrow \infty$. In view of this, we need to impose a physical constraint so that the resolvent curves in the k -cut critical points should have k cuts around $\zeta \rightarrow \infty$. This constraint is expressed in terms of *Stokes multipliers* for the *possible Stokes phenomena* in this system.

The second requirement, the non-perturbative stability of perturbative backgrounds, is imposed in the other formulation which is closely related to the Baker-Akhiezer function system: the so-called the Riemann-Hilbert (or inverse monodromy) approach [84–86] [23]. A brief flowchart of this approach is shown in Fig. 1. Details are given in Section 5, but in order to show how the Riemann-Hilbert approach works in resolving the issue, we here show the leading expression of the free energy (more precisely the two-point function of cosmological constant t) in the two-cut (1, 2) case:

$$\frac{\partial^2 \mathcal{F}(t; g_{\text{str}})}{\partial t^2} = [f(t)]^2, \quad f(t) = \sum_n s_{n,2,1} \int_{\mathcal{K}_n} \frac{d\lambda}{2\pi i} e^{g^{(2)}(t;\lambda) - g^{(1)}(t;\lambda)} + \dots \quad (1.9)$$

The parameter $s_{n,2,1}$ is a *Stokes multiplier* of the Baker-Akhiezer function system of the corresponding integrable system and the contour \mathcal{K}_n is an *anti-Stokes line* corresponding to the Stokes multiplier $s_{n,2,1}$. As one can suspect from the expression, the Riemann-Hilbert approach is directly related to the study of *Stokes phenomena* at $\zeta \rightarrow \infty$ in the ordinary differential equation of the Baker-Akhiezer system.

In this expression, the function $g^{(j)}(t; \zeta)$ is an *arbitrary function* but should be properly chosen so that the integrals other than the “leading” expression shown in Eq. (1.9) are

⁵The asymptotic expansion of Airy function is reviewed in Appendix A.

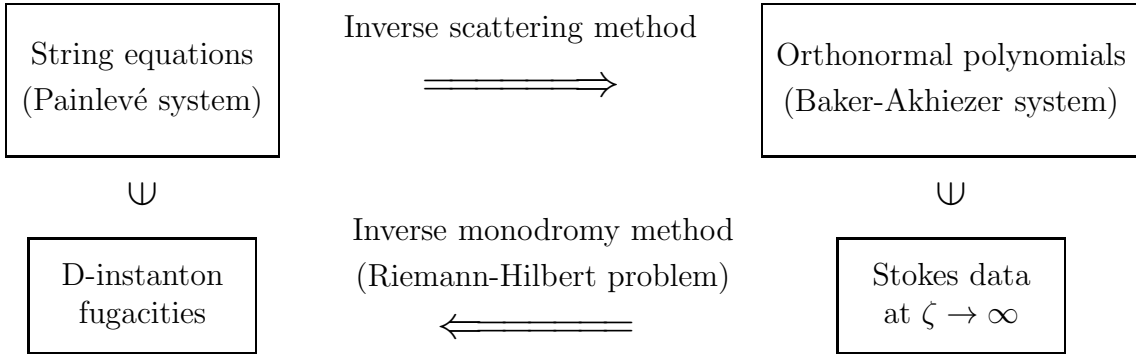


Figure 1: The Riemann Hilbert approach and the D-instanton chemical potentials (or fugacities)

negligible [86]. From the matrix-model viewpoints (to be discussed in Section 5), this function can be interpreted as *an off-shell string background geometry of string theory*. Therefore, if one *chooses* $g^{(j)}(t; \zeta)$ as a *macroscopic loop amplitude* realized in the large N limit of the matrix models, then the leading integral (1.9) becomes a similar expression to the mean field expression for a single eigenvalue of the matrix integral which appears in various studies in literature [28, 31, 32, 60, 61, 63].⁶ Therefore, the Stokes multipliers $s_{n,2,1}$ in Eq. (1.9) are directly identified as *the D-instanton chemical potentials* in the semi-classical saddle-point analysis. That is, *the first constraint is directly related to the constraint on the D-instanton chemical potentials*. Furthermore, since the Riemann-Hilbert integral, Eq. (1.9), provides the complete integration representation based on the reference string background $g^{(j)}(t; \zeta)$, we can discuss *non-perturbative stability of the background* $g^{(j)}(t; \zeta)$, especially for the background which is obtained as large N limit of the matrix models. This consideration for the stability is also expressed as a constraint on the Stokes multipliers and therefore the D-instanton chemical potentials. Originally, the mean field analyses include ambiguity of *choice of contour* and *weight of these contours* [28] and this fact becomes a cause of the ambiguity about the D-instanton chemical potentials in continuum loop-equation systems. In the Riemann-Hilbert approach, however, these degrees of freedom are identified as anti-Stokes lines \mathcal{K}_n and Stokes multipliers $s_{n,2,1}$, and they are tightly related to each other. As a consequence, the physical section of the D-instanton chemical potentials is obtained in the name of *non-perturbative completion*. This viewpoint is important in non-critical string theory because non-critical strings are sometimes defined as the large N (i.e. perturbative) expansion of unstable matrix-model critical points (e.g. (2, 3) bosonic minimal string theory) and therefore the matrix-model description does not necessary guarantee non-perturbative completion of string theory.⁷

As we will see in the coming sections, the above procedures completely determine the D-instanton chemical potentials in the two-cut (1, 2) critical points and results in the Hastings-McLeod solution [89] to the Painlevé II equation (in Section 5.1). Actually it is known that this is the unique solution which realizes the two phases of the two-cut (1, 2)

⁶It is interesting that the Riemann-Hilbert expression gives a similar expression to the D-instanton operators obtained in the free-fermion formulation [31, 32].

⁷Early investigations of non-perturbative complete string theories are found in [16, 36, 87, 88].

critical point of the two-cut matrix model,⁸ and therefore the Hastings-McLeod solution is suitable for this critical point. An advantage of our work is the discovery of the actual physical requirements to obtain the correct solutions to the non-perturbative completion which are also applicable to the critical points with an arbitrary number of cuts.

The next main developments shown in this paper is, therefore, an extension of our procedure to the general multi-cut cases (which even reaches to ∞ -cut!). In particular, general structures of Stokes multipliers in the $k \times k$ isomonodromy systems are investigated in Section 3, and a new way to identify non-trivial Stokes multipliers is proposed (Theorem 4) with terminology of *the profile of dominant exponents*. Furthermore, explicit solutions are obtained with help of the physical constraints, i.e. the multi-cut boundary conditions (Theorem 7 and 8 in Section 4.3). In this sense, our solutions provide the multi-cut generalization of the Hastings-McLeod solution. Interestingly, we found that these solutions are labeled by constrained Young diagrams (Proposition 3 in Section 4.3). This result implies that there is a quite rich world beyond the non-perturbative horizon, and that the multi-cut matrix models provide fruitful fields for a quantitative study of these issues.

Organization of this paper is as follows: In Section 2, after summarizing the asymptotic expansion of the ODE system in the multi-cut critical points, the general facts about Stokes phenomenon in ordinary differential equations are reviewed. As a warming up, the case of the two-cut $(1, 2)$ critical point is also shown. In Section 3, Stokes phenomena in the multi-cut critical points are studied. In particular, a systematic way of reading the Stokes multipliers in general cases is developed. In Section 4, the multi-cut boundary condition is proposed. In Section 4.3, the discrete and continuum solutions are shown. In Section 5, the non-perturbative stability condition is studied in terms of the Riemann-Hilbert problem. Section 6 is devoted to conclusion and discussion.

Context of Appendices is: Appendix A is about the Stokes phenomenon of Airy function (a review of [52]). Appendix B is about calculation of Lax operators. Appendix C is about supplements to Theorem 5 and Theorem 6 with some examples of the multi-cut boundary-condition recursive equations. Appendix D is about derivation of continuum solutions. Appendix E is about calculation of the 3-cut $(1, 1)$ critical points and Appendix F is about calculation of 4-cut $(1, 1)$ critical points.

2 Stokes phenomena in the ODE systems

Before we devote ourselves into the multi-cut systems, here we first review some general facts about Stokes phenomenon in ordinary differential equation systems, then we summarize the well-studied two-cut $(1, 2)$ case. This two-cut system has been exten-

⁸It was shown by Hastings-McLeod [89] that their solution is a unique solution to the Painlevé II equation, Eq. (2.43), which realizes the following asymptotic behaviors of $f(t)$ on the two sides of infinity $t \rightarrow \pm\infty$:

$$\frac{1}{2}f''(t) - f^3(t) + 2tf(t) = 0 : \quad f(t \rightarrow \infty) \sim 0, \quad f(t \rightarrow -\infty) \sim \sqrt{-t}, \quad (1.10)$$

which is the same behavior as the two-cut $(1, 2)$ critical point of the two-cut matrix model discussed in [42]. For some mathematical derivation of this solution in the two-cut matrix models, see also [90] which has been studied within the Riemann-Hilbert problem.

sively studied not only in physical context [34–39, 42, 53, 69] but also in mathematical context [85, 86, 89, 92–95], since it is related to the Hastings-McLeod solution [89] of the Painlevé II system. For more comprehensive and rigorous reviews and references on the isomonodromy deformations, Stokes phenomenon and inverse monodromy problems, see [91]. We also note that the idea of isomonodromy deformation was introduced in non-critical string theory by [23].

2.1 The ODE system and asymptotic expansions

It was first proposed in [69] that the multi-cut matrix models are controlled by multi-component KP hierarchy [96] and therefore by the following Baker-Akhiezer function system:

$$\zeta \Psi(t; \zeta) = \mathbf{P}(t; \partial) \Psi(t; \zeta), \quad (2.1)$$

$$g_{\text{str}} \frac{\partial}{\partial \zeta} \Psi(t; \zeta) = \mathbf{Q}(t; \partial) \Psi(t; \zeta). \quad (2.2)$$

Here the operator $\mathbf{P}(t; \partial)$ and $\mathbf{Q}(t; \partial)$ are \hat{p} -th and \hat{q} -th order differential operators in $\partial \equiv g_{\text{str}} \partial_t$, respectively, which satisfy the Douglas (string) equation [21]:

$$[\mathbf{P}(t; \partial), \mathbf{Q}(t; \partial)] = g_{\text{str}} I_k. \quad (2.3)$$

Critical points in the multi-cut two-matrix models are characterized by these Lax operators and explicitly obtained in [46] with their critical potentials. There are two kinds of interesting critical points: the \mathbb{Z}_k -symmetric critical points and fractional-superstring critical points. A brief summary of the corresponding Baker-Akhiezer function system is following:⁹

1. The \mathbb{Z}_k -symmetric critical points are characterized by the following $k \times k$ Lax operators [46]:

$$\mathbf{P}(t; \partial) = \Gamma \partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(Z_k P)}(t) \partial^n, \quad \mathbf{Q}(t; \partial) = \Gamma^{-1} \partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(Z_k Q)}(t) \partial^n, \quad (2.4)$$

with the shift matrix Γ ,

$$\Gamma = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad (2.5)$$

and the $k \times k$ matrix-valued real coefficients $U_n^{(Z_k P)}(t)$ and $U_n^{(Z_k Q)}(t)$ which satisfy

$$U_n^{(Z_k P)}(t) = \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & \ddots & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix}, \quad U_n^{(Z_k Q)}(t) = \begin{pmatrix} 0 & & & & * \\ * & 0 & & & \\ & \ddots & \ddots & & \\ & & * & 0 & \\ & & & * & 0 \end{pmatrix}, \quad (2.6)$$

⁹For the derivation of these systems from the multi-cut two-matrix models, see [46].

as a result of the \mathbb{Z}_k symmetry of the critical points. Macroscopic loop amplitudes (i.e. off-critical resolvent amplitudes with $t \neq 0$) in this kind of critical points are also obtained in [46] with the Daul-Kazakov-Kostov prescription [30] and expressed as the Jacobi polynomials or the third and fourth Chebyshev polynomials. In particular, the amplitudes in the k -cut $(1, 1)$ critical points are given as the eigenvalues of the Lax operators Eq. (2.4) in the weak coupling limit $g_{\text{str}} \rightarrow 0$:¹⁰

$$\begin{aligned}\mathbf{P}(t; \partial) &\simeq \text{diag}_{j=1}^k \left(P_{\text{classical}}^{(j)}(t; z) \right) = \text{diag}_{j=1}^k \left(\omega^{j-1} x(z) \right), \\ \mathbf{Q}(t; \partial) &\simeq \text{diag}_{j=1}^k \left(Q_{\text{classical}}^{(j)}(t; z) \right) = \text{diag}_{j=1}^k \left(\omega^{-(j-1)} y(z) \right),\end{aligned}\quad (2.7)$$

with

$$x(z) = t \sqrt[k]{(z-c)^l (z-b)^{k-l}}, \quad y(z) = t \sqrt[k]{(z-c)^{k-l} (z-b)^l} \quad (2.8)$$

and $0 = cl + b(k-l)$ and the dimensionless variable $z \equiv g_{\text{str}} t^{-1} \partial_t$.

2. The fractional-superstring critical points [43] are characterized by the following two kinds of Lax operators [46]: The first kind is given as

$$\mathbf{P}(t; \partial) = \Gamma \partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(F_k P)}(t) \partial^n, \quad \mathbf{Q}(t; \partial) = \Gamma \partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(F_k Q)}(t) \partial^n. \quad (2.9)$$

These Lax operators are derived from the $\omega^{1/2}$ -rotated critical potentials. The second kind is given as

$$\mathbf{P}(t; \partial) = \Gamma^{(\text{real})} \partial^{\hat{p}} + \sum_{n=0}^{\hat{p}-1} U_n^{(R_k P)}(t) \partial^n, \quad \mathbf{Q}(t; \partial) = \Gamma^{(\text{real})} \partial^{\hat{q}} + \sum_{n=0}^{\hat{q}-1} U_n^{(R_k Q)}(t) \partial^n, \quad (2.10)$$

with the matrix $\Gamma^{(\text{real})}$,

$$\Gamma^{(\text{real})} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -1 & & & & 0 \end{pmatrix}. \quad (2.11)$$

These Lax operators are derived from the real critical potentials. In both cases, all the $k \times k$ matrix-valued coefficients $U_n^{(F_k P)}(t)$ and $U_n^{(F_k Q)}(t)$ (and $U_n^{(R_k P)}(t)$ and $U_n^{(R_k Q)}(t)$) are real functions. The macroscopic loop amplitudes in each case are obtained and given by the deformed Chebyshev functions [44].

In this paper, for the sake of simplicity, we concentrate on the $\hat{p} = 1$ cases of the \mathbb{Z}_k -symmetric critical points. With this choice of critical points, the Lax operator $\mathbf{P}(t; \partial)$ becomes

$$\mathbf{P}(t; \partial) = \Gamma \partial + H(t), \quad (2.12)$$

¹⁰In this paper, the equality \simeq means that they are equal up to some similarity transformation.

and the Baker-Akhiezer function for the eigenvalue problem of the operator $\mathbf{P}(t; \partial)$, Eq. (2.1), is rewritten as

$$g_{\text{str}} \frac{\partial}{\partial t} \Psi(t; \zeta) = \mathcal{P}(t; \zeta) \Psi(t; \zeta) \equiv \Gamma^{-1} [\zeta - H(t)] \Psi(t; \zeta), \quad (2.13)$$

and therefore Eq. (2.2) is also rewritten as a $k \times k$ matrix polynomial operator in ζ :

$$g_{\text{str}} \frac{\partial \Psi(t; \zeta)}{\partial \zeta} = \mathcal{Q}(t; \zeta) \Psi(t; \zeta) \equiv \mathbf{Q}(t; \partial) \Psi(t; \zeta), \quad \mathcal{Q}(t; \zeta) = \sum_{n=1}^r \mathcal{Q}_{-n}(t) \zeta^{n-1}. \quad (2.14)$$

Here we define r as

$$r \equiv \hat{q} + 1 > 0, \quad (2.15)$$

which is referred to as the Poincaré index in literature. The advantage of this formulation is that the pair of Lax operators $(\mathbf{P}(t; \partial), \mathbf{Q}(t; \partial))$ becomes a pair of the polynomial operators $(\mathcal{P}(t; \zeta), \mathcal{Q}(t; \zeta))$, and the system can be expressed as an $k \times k$ first order ordinary differential equation (ODE) system. These systems are called the Zakharov-Shabat eigenvalue problem [97] or AKNS hierarchy [98] in literature. Note that the Douglas equation becomes

$$[\mathbf{P}(t; \partial), \mathbf{Q}(t; \partial)] = g_{\text{str}} I_k \Leftrightarrow [g_{\text{str}} \partial_\zeta - \mathcal{Q}(t; \zeta), g_{\text{str}} \partial_t - \mathcal{P}(t; \zeta)] = 0, \quad (2.16)$$

in terms of these Lax operators.

This ODE system Eq. (2.14) has the k independent order k column vector solutions $\Psi^{(j)}(t; \zeta)$, ($j = 1, 2, \dots, k$), and we here use the following matrix solution notation:

$$\Psi(t; \zeta) \equiv \left(\Psi^{(1)}(t; \zeta), \dots, \Psi^{(k)}(t; \zeta) \right). \quad (2.17)$$

As in the usual ODE, we consider formal expansion around $\zeta \rightarrow \infty$. However the point $\zeta \rightarrow \infty$ is an irregular singularity and the formal series expansion around this irregular point in general does not converge absolutely. Up to proper redefinition of the k independent solutions, the formal series expansion of the solutions around $\zeta \rightarrow \infty$ is given as

$$\Psi_{\text{asym}}(t; \zeta) \equiv Y(t; \zeta) e^{\frac{1}{g_{\text{str}}} \varphi(t; \zeta)} \equiv \left[I_k + \sum_{n=1}^{\infty} \frac{Y_n(t)}{\zeta^n} \right] \times \exp \left[\frac{1}{g_{\text{str}}} \left(\varphi_0 \ln \zeta - \sum_{m=-r, \neq 0}^{\infty} \frac{\varphi_m(t)}{m \zeta^m} \right) \right]. \quad (2.18)$$

The coefficient matrices are obtained from the recursive equations,

$$0 = -n g_{\text{str}} Y_n(t) + \sum_{m=0}^{n+r} \left[Y_m(t) \varphi_{n-m}(t) - \mathcal{Q}_{n-m}(t) Y_m(t) \right], \quad (n = -r, -r+1, \dots). \quad (2.19)$$

For convenience, we extend the indices of the coefficient matrices:

$$Y_0(t) = I_k, \quad Y_n(t) = 0 \quad (n < 0), \quad \varphi_m(t) = \mathcal{Q}_m(t) = 0 \quad (m < -r), \quad (2.20)$$

and impose the following constraints on $Y_n(t)$ and $\varphi_n(t)$:

$$[\Gamma^l, \varphi_n(t)] = 0, \quad \sum_{i=1}^k [Y_n(t)]_{i, i+l} = 0, \quad (l = 0, 1, \dots, k-1). \quad (2.21)$$

This recursive relation then can be solved uniquely and all the expansion coefficient are written with the coefficient matrix-valued function $H(t)$ in Eq. (2.12).

On the other hand, it is also convenient to use a diagonal basis, $\tilde{\Psi}_{asym}(t; \zeta)$, which is defined by

$$\begin{aligned} \tilde{\Psi}_{asym}(t; \zeta) &\equiv \tilde{Y}(t; \zeta) e^{\frac{1}{g_{str}} \tilde{\varphi}(t; \zeta)} \equiv \left[I_k + \sum_{n=1}^{\infty} \frac{\tilde{Y}_n(t)}{\zeta^n} \right] \times \exp \left[\frac{1}{g_{str}} \left(\tilde{\varphi}_0 \ln \zeta - \sum_{m=-r, \neq 0}^{\infty} \frac{\tilde{\varphi}_m(t)}{m \zeta^m} \right) \right] \\ &\equiv U^\dagger \Psi_{asym}(t; \zeta) U, \end{aligned} \quad (2.22)$$

where the matrix U is given as

$$U_{jl} = \frac{1}{\sqrt{k}} \omega^{(j-1)(l-1)}, \quad \Gamma U = U \Omega, \quad (2.23)$$

with $\Omega = \text{diag}(1, \omega, \omega^2, \dots, \omega^{k-1})$ and $\omega = e^{2\pi i/k}$. Since this is a similarity transformation, the coefficients also satisfy the same recursive relation (2.19). In this basis, the function $\tilde{\varphi}(t; \zeta)$ is a diagonalized matrix and we write its eigenvalues as

$$\tilde{\varphi}(t; \zeta) = \text{diag}(\varphi^{(1)}(t; \zeta), \dots, \varphi^{(k)}(t; \zeta)). \quad (2.24)$$

The vector components of the formal series, $\tilde{\Psi}_{asym} = (\tilde{\Psi}_{asym}^{(1)}, \dots, \tilde{\Psi}_{asym}^{(k)})$, is given as

$$\tilde{\Psi}_{asym}^{(j)}(t; \zeta) = \tilde{Y}^{(j)}(t; \zeta) e^{\frac{1}{g_{str}} \varphi^{(j)}(t; \zeta)}, \quad (j = 1, 2, \dots, k), \quad (2.25)$$

with $\tilde{Y}(t; \zeta) = (\tilde{Y}^{(1)}, \dots, \tilde{Y}^{(k)})$.

Although the above formal solutions are formal series around the irregular singularity, they are related to the exact analytic solutions of the ODE system, $\tilde{\Psi}(t; \zeta)$, in the sense of *asymptotic expansion*:

$$\tilde{\Psi}(t; \zeta) \underset{asym}{\simeq} \tilde{\Psi}_{asym}(t; \zeta) C, \quad (2.26)$$

in some specific angular domain [99]:

$$\zeta \rightarrow \infty \in D(a, b) \equiv \{\zeta \in \mathbb{C}; a < \arg(\zeta) < b\}. \quad (2.27)$$

An example of the angular domain is shown in Fig. 2-a. Here C is a proper coefficient matrix, and the meaning of asymptotic expansion is following:

Definition 1 (asymptotic expansion) For a holomorphic function $f(\zeta)$, an asymptotic expansion of $f(\zeta)$ in a domain $D(a, b)$ is defined as a formal series $\sum_n f_n \zeta^{-n}$ such that there exists a constant $B_{R; a, b}^{(N)} \in \mathbb{R}$ which satisfies

$$\left| f(\zeta) - \sum_{n=-r}^N \frac{f_n}{\zeta^n} \right| < \frac{B_{R; a, b}^{(N)}}{|\zeta|^N}, \quad \zeta \in D(a, b) \cap \{\zeta \in \mathbb{C}; |\zeta| > R\} \quad (2.28)$$

for each integer $N = -r, -r+1, \dots$ and sufficiently large $R \in \mathbb{R}$. This is written as

$$f(\zeta) \underset{asym}{\simeq} \sum_{n=-r}^{\infty} \frac{f_n}{\zeta^n}, \quad \zeta \rightarrow \infty \in D(a, b). \quad (2.29)$$

The maximal angular domains are called *Stokes sectors*.

2.2 General facts on Stokes phenomena in the ODE system

In this subsection, in order to understand the asymptotic expansion Eqs. (2.22) and (2.26), we review some general theorem about the asymptotic expansions and Stokes phenomena in the general $k \times k$ ODE systems,

$$\begin{aligned} g_{\text{str}} \frac{\partial}{\partial \zeta} \tilde{\Psi}(t; \zeta) &= \left[\tilde{\mathcal{Q}}_{-r} \zeta^{r-1} + \tilde{\mathcal{Q}}_{-r+1}(t) \zeta^{r-2} + \cdots \tilde{\mathcal{Q}}_{-1}(t) \right] \tilde{\Psi}(t; \zeta) \\ &\equiv \tilde{\mathcal{Q}}(t; \zeta) \tilde{\Psi}(t; \zeta). \end{aligned} \quad (2.30)$$

Note that proof of the theorems appearing in this subsection can be found in [91] and references therein. For sake of simplicity, we assume

$$\tilde{\mathcal{Q}}_{-r} = \text{diag}(A_1, A_2, \dots, A_k), \quad A_i - A_j \neq 0, \quad A_i \neq 0, \quad (i, j = 1, 2, \dots, k). \quad (2.31)$$

Therefore, the exponents Eq. (2.24) are expressed as

$$\tilde{\varphi}(t; \zeta) = \tilde{\varphi}_0(t) \ln \zeta - \sum_{n=-r, n \neq 0}^{\infty} \frac{\tilde{\varphi}_n(t)}{n \zeta^n} = \frac{1}{r} \tilde{\mathcal{Q}}_{-r} \zeta^r + \cdots, \quad (2.32)$$

and $\varphi_{-r}^{(i)} = A_i$ ($i = 1, 2, \dots, k$) also satisfies (2.31).

The meaning of the asymptotic expansion Eq. (2.22) is that basically we ignore relatively small exponents. One takes some (small enough) angular domain $D(a, e^{i\epsilon}a)$ then compares the relative magnitudes around $\zeta \rightarrow \infty$, for example,

$$|e^{\varphi^{(j_1)}(t; \zeta)}| < |e^{\varphi^{(j_2)}(t; \zeta)}| < \cdots < |e^{\varphi^{(j_k)}(t; \zeta)}|, \quad \zeta \rightarrow \infty \in D(a, e^{i\epsilon}a). \quad (2.33)$$

Then one can obtain the following equality under the asymptotic expansion:

$$e^{\varphi^{(j_2)}(t; \zeta)} + \theta e^{\varphi^{(j_1)}(t; \zeta)} \underset{\text{asym}}{\simeq} e^{\varphi^{(j_2)}(t; \zeta)}, \quad \zeta \rightarrow \infty \in D(a, e^{i\epsilon}a). \quad (2.34)$$

That is, the smaller exponents become practically invisible in view of the asymptotic expansion. Our interest is to identify angles of ζ where the the exponents, $\exp(\varphi^{(j)}(\zeta))$ ($i = 1, 2, \dots, k$), change the relative magnitudes around $\zeta \rightarrow \infty$. This leads to the concept of *Stokes lines*:

Definition 2 (Stokes lines) *With the assumption (2.31), Stokes lines $\text{SL}_{j,l}$ in this ODE system are defined for each pair of (j, l) as*

$$\text{SL}_{j,l} \equiv \left\{ \zeta \in \mathbb{C}; \text{Re}[(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)})\zeta^r] = 0 \right\} = \bigcup_{n=0}^{2r-1} \text{SL}_{j,l}^{(n)}, \quad (2.35)$$

which consists of $2r$ semi-infinite lines, $\text{SL}_{j,l}^{(n)}$ ($n = 0, 1, \dots, 2r-1$). The set of lines, SL , denotes a set of whole Stokes lines, $\text{SL} \equiv \bigcup_{j,l} \text{SL}_{j,l}$.

An example of Stokes lines $\text{SL}_{j,l}$ is shown in Fig. 2-b. In particular, if the angular domain $D(a, b)$ of the asymptotic expansion includes a Stokes line, one cannot neglect the exponents as it happens in Eq. (2.34). This leads to the following definition of *Stokes sectors*:

Definition 3 (Stokes sectors) A Stokes sector D in the ODE system is an angular domain, $D = D(a, b)$, with angles a and b such that for each pair of (j, l) there exist a unique Stokes line $SL_{j,l}^{(n_{j,l})}$ which satisfies,

$$SL_{j,l}^{(n_{j,l})} \subset D = D(a, b), \quad (2.36)$$

that is, except for this line $SL_{j,l}^{(n_{j,l})}$ there is no other line $SL_{j,l}^{(n'_{j,l})}$ ($\neq SL_{j,l}^{(n_{j,l})}$) which runs inside the domain, D .

An example of the Stokes sectors (the 3-cut $(1, 1)$ critical point) is shown in Fig. 2-b.

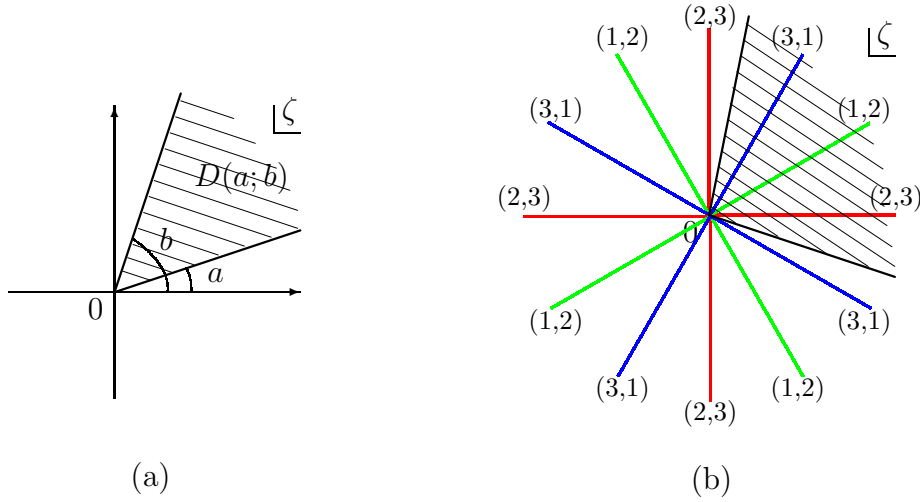


Figure 2: a) An angular domain of $D(a, b)$. b) Stokes lines and Stokes sectors. This is the 3-cut $(1, 1)$ critical points. An example of Stokes sectors is also shown. In this critical point, there are three kinds of the Stokes lines $SL_{i,j}$, $(i, j) = (1, 2), (2, 3), (3, 1)$. Stokes sectors includes one and only one Stokes line of each kind.

Actually the definition of the Stokes sectors results in the following theorem [99]:

Theorem 1 For a given Stokes sector D , any solutions to the ODE system $\tilde{\Psi}(t; \zeta)$ has the following asymptotic expansion:

$$\tilde{\Psi}(t; \zeta) \underset{asym}{\simeq} \tilde{\Psi}_{asym}(t; \zeta) C, \quad \zeta \rightarrow \infty \in D, \quad (2.37)$$

with a matrix C . Furthermore, the coefficient matrix C (i.e. asymptotic expansion) is unique in the Stokes sector D .

This uniqueness enables us to define the following unique solution in a Stokes sector D :

Definition 4 (Canonical solution) If the solution to the ODE system, $\tilde{\Psi}_{can}(t; \zeta)$, has the asymptotic expansion with $C = I_k$ in a Stokes sector D ,

$$\tilde{\Psi}_{can}(t; \zeta) \underset{asym}{\simeq} \tilde{\Psi}_{asym}(t; \zeta), \quad \zeta \rightarrow \infty \in D, \quad (2.38)$$

this solution is called the canonical solution in the Stokes sector D .

This theorem on the other hand means that the asymptotic expansion is not unique if one chooses some angular domain D' narrower than Stokes sectors. In particular, as is shown in Fig. 3, the intersection of two different Stokes sectors D_1 and D_2 is generally narrower than Stokes sectors, and therefore there appears difference between the canonical solutions $\tilde{\Psi}_i(t; \zeta)$ of each sector $D_i (i = 1, 2)$:

$$\tilde{\Psi}_2(t; \zeta) = \tilde{\Psi}_1(t; \zeta) S, \quad D_1 \cap D_2 \neq \emptyset. \quad (2.39)$$

This $k \times k$ matrix S which expresses the difference between $\tilde{\Psi}_1(t; \zeta)$ and $\tilde{\Psi}_2(t; \zeta)$ is called a *Stokes matrix* in the intersection $D_1 \cap D_2$. This indicates that solutions in the ODE system generally have different asymptotic expansion in different Stokes sectors. This analytic behavior of the solutions is referred to as the *Stokes phenomenon* in the ODE system.

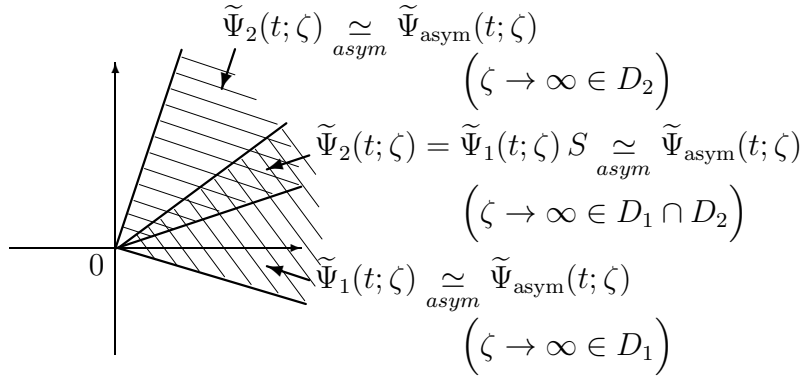


Figure 3: Explanation of Stokes phenomenon in ODE systems. For given two Stokes sectors, their canonical solutions are generally different by a Stokes matrix, S in the intersection $D_1 \cap D_2$. This behavior of analytic functions is called Stokes phenomenon.

A direct calculation shows that the Stokes matrices do not depend on ζ , and furthermore, they do not depend on the deformation parameter t either (as in (2.13)):

$$\frac{dS}{d\zeta} = \frac{dS}{dt} = 0. \quad (2.40)$$

This means that the Stokes matrices are understood as integration constants for the evolution system in the t space. Therefore, these integrable deformations in the original multi-component KP hierarchy are also called *isomonodromy deformation system* [84]. This also leads us to the concept of *inverse monodromy approach* [84, 85], which is also briefly reviewed in Section 5.

Components of Stokes matrices satisfy the following theorem (See [91], for example):

Theorem 2 (Stokes multipliers) *For given two Stokes sectors D_1 and D_2 ($D_1 \cap D_2 \neq \emptyset$), components of their Stokes matrices, i.e. Stokes multipliers, $S = (s_{i,j})$, satisfy*

$$s_{j,j} = 1 \quad (j = 1, 2, \dots, k), \quad (2.41)$$

and $s_{i,j}$ ($i \neq j$) can take non-zero values only when the exponents satisfy

$$\operatorname{Re}[\varphi_{-r}^{(i)}\zeta^r] < \operatorname{Re}[\varphi_{-r}^{(j)}\zeta^r] \text{ for all angular range of } \zeta \rightarrow \infty \in D_1 \cap D_2 \neq \emptyset. \quad (2.42)$$

In particular, these Stokes multipliers, $s_{i,j}$ ($i \neq j$), are called “non-trivial”.

In this paper, we often refer to these facts about Stokes phenomena in ODE systems.

2.3 Stokes phenomena in the two-cut case

In this subsection, we specialize the general consideration to the two-cut $(1, 2)$ critical point.

2.3.1 The ODE system and asymptotic expansions in the two-cut case

The string equation in this system is known as the Painlevé II equation [35, 36],

$$\frac{g_{\text{str}}^2}{2} \ddot{f} - f^3 + 2tf = 0, \quad (2.43)$$

which is equivalent to the following ODE system in ζ (Eq. (2.45)) with its isomonodromy deformations in t (Eq. (2.46)):¹¹

$$g_{\text{str}} \frac{\partial}{\partial \zeta} \tilde{\Psi}(t; \zeta) = \left[\sigma_3 \zeta^2 - (\sigma_1 f) \zeta + \left(-\frac{1}{2} f^2 + \mu \right) \sigma_3 - g_{\text{str}} \frac{i}{2} \sigma_2 \dot{f} \right] \tilde{\Psi}(t; \zeta), \quad (2.45)$$

$$g_{\text{str}} \frac{\partial}{\partial t} \tilde{\Psi}(t; \zeta) = \left[\sigma_3 \zeta - \sigma_1 f(t) \right] \tilde{\Psi}(t; \zeta). \quad (2.46)$$

Since this 2×2 first-order ODE system has two independent column vector solutions $\tilde{\Psi}^{(1)}(t; \zeta)$ and $\tilde{\Psi}^{(2)}(t; \zeta)$, we use the matrix notation for the solutions:

$$\tilde{\Psi}(t; \zeta) = \left(\tilde{\Psi}^{(1)}(t; \zeta), \tilde{\Psi}^{(2)}(t; \zeta) \right). \quad (2.47)$$

At the point $\zeta \rightarrow \infty$, the ODE has an irregular singularity (of the Poincaré order 3) and the formal expansion of the solutions (2.22) is given as

$$\begin{aligned} \tilde{\Psi}_{\text{asym}}(\zeta; t) &= \left[I_2 + \frac{i}{2\zeta} \sigma_2 f(t) + \mathcal{O}(1/\zeta^2) \right] \exp \left[\frac{1}{g_{\text{str}}} \left(\frac{1}{3} \sigma_3 \zeta^3 + \mu \sigma_3 \zeta + \mathcal{O}(1/\zeta) \right) \right] \\ &\equiv \tilde{Y}(t; \zeta) e^{\frac{1}{g_{\text{str}}} \tilde{\varphi}(t; \zeta)}. \end{aligned} \quad (2.48)$$

This can be obtained with the recursion relation Eq. (2.19) (see also in Appendix B.2). Note that the exponent $\tilde{\varphi}(t; \zeta)$ is a diagonal matrix which satisfies $\tilde{\varphi}(t; \zeta) \propto \sigma_3$, and then each vector solution $\tilde{\Psi}_{\text{asym}}^{(i)}(t; \zeta)$ ($i = 1, 2$) has different exponents:

$$\tilde{\Psi}_{\text{asym}}^{(i)}(t; \zeta) = \tilde{Y}^{(i)}(t; \zeta) e^{\frac{1}{g_{\text{str}}} \varphi^{(i)}(t; \zeta)}, \quad (2.49)$$

with

$$\tilde{Y}(t; \zeta) = (\tilde{Y}^{(1)}(t; \zeta), \tilde{Y}^{(2)}(t; \zeta)), \quad \tilde{\varphi}(t; \zeta) = \operatorname{diag}(\varphi^{(1)}(t; \zeta), \varphi^{(2)}(t; \zeta)). \quad (2.50)$$

¹¹In the later discussion (from Section 3), we also define a different basis: $\Psi(t; \zeta) \equiv U \tilde{\Psi}(t; \zeta) U^\dagger$, with

$$U \sigma_3 U^\dagger = \sigma_1, \quad U \sigma_1 U^\dagger = -\sigma_3, \quad U \sigma_2 U^\dagger = \sigma_2. \quad (2.44)$$

This basis naturally appears in the matrix-model calculations and is more suitable to read the Hermiticity of the multi-cut matrix models [46].

2.3.2 Stokes sectors and Stokes matrices

In this case, there is only one kind of the Stokes lines $SL_{1,2}$ which is given by (2.35) as

$$\zeta = |\zeta|e^{i\theta} : \quad \theta = \frac{\pi}{6} + \frac{n\pi}{3} \quad (n = 0, 1, \dots, 5). \quad (2.51)$$

Therefore, Stokes sectors D_n are given as

$$D_n = e^{ni\frac{\pi}{3}} D_0, \quad (n = 0, 1, \dots, 5), \quad D_0 \equiv D\left(-\frac{\pi}{2}, \frac{\pi}{6}\right). \quad (2.52)$$

This is shown in Fig. 4. The canonical solution on the Stokes sector D_n is denoted by $\tilde{\Psi}_n(t; \zeta)$. The Stokes matrices S_n are now defined as

$$S_n \equiv \tilde{\Psi}_n^{-1}(t; \zeta) \tilde{\Psi}_{n+1}(t; \zeta), \quad (n = 0, 1, \dots, 5), \quad (2.53)$$

and therefore components of the Stokes matrices are read as

$$\begin{aligned} D_{2n} \cap D_{2n+1} : \quad S_{2n} &= \begin{pmatrix} 1 & 0 \\ s_{2n} & 1 \end{pmatrix}; \quad \left(|e^{\varphi^{(1)}(t; \zeta)}| > |e^{\varphi^{(2)}(t; \zeta)}|, \quad \zeta \rightarrow \infty \right), \\ D_{2n+1} \cap D_{2n+2} : \quad S_{2n+1} &= \begin{pmatrix} 1 & s_{2n+1} \\ 0 & 1 \end{pmatrix}; \quad \left(|e^{\varphi^{(1)}(t; \zeta)}| < |e^{\varphi^{(2)}(t; \zeta)}|, \quad \zeta \rightarrow \infty \right). \end{aligned} \quad (2.54)$$

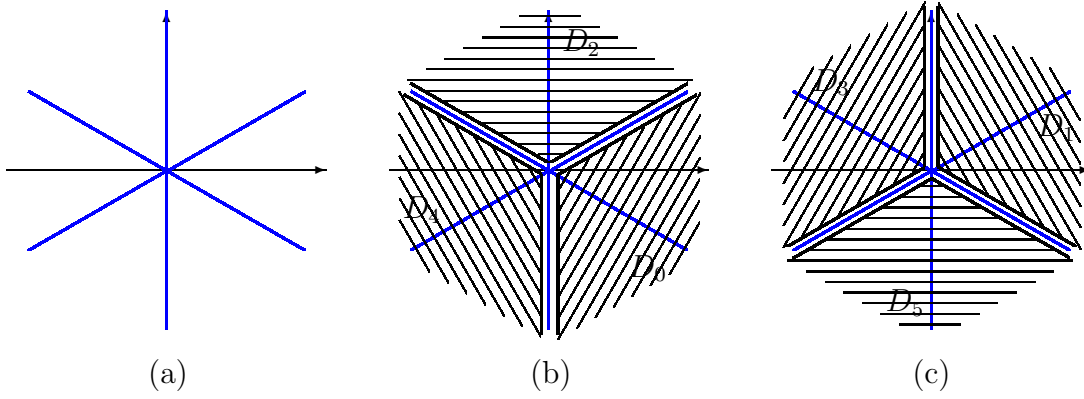


Figure 4: a) Stokes lines in the two-cut (1,2) case. b) Stokes sectors of D_0, D_2 and D_4 . c) Stokes sectors of D_1, D_3 and D_5 .

2.3.3 Three basic constraints on the Stokes multipliers

The Stokes multipliers satisfy three constraints from the symmetry of the original ODE system.

\mathbb{Z}_2 symmetry constraint This symmetry originates from the \mathbb{Z}_2 symmetry of the matrix model. That is, this is the reflection symmetry $M \rightarrow -M$ of the one-matrix models:

$$\mathcal{Z} = \int dM e^{-N \text{tr} V(M)}, \quad V(-M) = V(M). \quad (2.55)$$

In terms of the ODE system, this symmetry is expressed by the reflection of $\zeta \rightarrow -\zeta$:

$$\begin{aligned} g_{\text{str}} \frac{\partial \tilde{\Psi}(t; -\zeta)}{\partial \zeta} &= [-\tilde{\mathcal{Q}}(t; -\zeta)] \tilde{\Psi}(t; -\zeta) = [\sigma_1 \tilde{\mathcal{Q}}(t; \zeta) \sigma_1] \tilde{\Psi}(t; -\zeta), \\ g_{\text{str}} \frac{\partial \tilde{\Psi}(t; -\zeta)}{\partial t} &= [\tilde{\mathcal{P}}(t; -\zeta)] \tilde{\Psi}(t; -\zeta) = [\sigma_1 \tilde{\mathcal{P}}(t; \zeta) \sigma_1] \tilde{\Psi}(t; -\zeta). \end{aligned} \quad (2.56)$$

Therefore, each canonical solution is mapped to another canonical solution as:

$$\sigma_1 \tilde{\Psi}_n(t; -\zeta) \sigma_1 = \tilde{\Psi}_{n+3}(t; \zeta), \quad (n = 0, 1, \dots, 5), \quad (2.57)$$

and the Stokes matrices are mapped as

$$S_{n+3} = \sigma_1 S_n \sigma_1, \quad s_{n+3} = s_n, \quad (n = 0, 1, \dots, 5). \quad (2.58)$$

Consequently there are only three independent Stokes multipliers,

$$s_0 = s_3 \equiv \alpha, \quad s_1 = s_4 \equiv \beta, \quad s_2 = s_5 = \gamma. \quad (2.59)$$

Hermiticity constraint This originates from Hermiticity of the matrix models. In the two-cut cases, they are studied in [39, 69]. This symmetry is expressed as¹²

$$\tilde{\mathcal{Q}}^*(t; \zeta^*) = \tilde{\mathcal{Q}}(t; \zeta^*), \quad \tilde{\mathcal{P}}^*(t; \zeta^*) = \tilde{\mathcal{P}}(t; \zeta^*). \quad (2.60)$$

Therefore, each canonical solution is mapped to another canonical solution as:

$$\tilde{\Psi}_n^*(t; \zeta^*) = \tilde{\Psi}_{7-n}(t; \zeta), \quad (n = 0, 1, \dots, 5), \quad (2.61)$$

and the Stokes matrices are mapped as

$$S_n^* = S_{6-n}^{-1}, \quad s_n^* + s_{6-n} = 0, \quad (n = 0, 1, \dots, 5). \quad (2.62)$$

This reduces three independent Stokes multipliers α, β and γ to be two real parameters:

$$\alpha^* + \alpha = 0, \quad \beta^* + \gamma = 0. \quad (2.63)$$

Monodromy free constraint The last constraint is the requirement that the solutions to the ODE system are single-valued functions. Note that the presence of non-trivial monodromy in the context of matrix models corresponds to introducing background RR flux and/or D0-branes in 0A string background. That is, the system becomes like the complex matrix models [42, 88, 101]. This constraint for the single-valued solutions is expressed as

$$\tilde{\Psi}_n(t; \zeta) = \tilde{\Psi}_n(t; e^{2\pi i} \zeta) = \tilde{\Psi}_{n+6}(t; \zeta), \quad (2.64)$$

therefore

$$S_0 S_1 S_2 S_3 S_4 S_5 = I_2, \quad (2.65)$$

¹²Note that we use the following convention of complex conjugation in this paper: $[f(\zeta)]^* = f^*(\zeta^*) = \sum_n f_n^* \zeta^*$, with a function $f(\zeta) \equiv \sum_n f_n \zeta^n$.

which results in

$$s_0 + s_1 + s_2 + s_0 s_1 s_3 = \alpha(1 - |\beta|^2) + \beta - \beta^* = 0. \quad (2.66)$$

Taking all constraints Eqs. (2.59), (2.63), (2.66) into consideration, we find that the Stokes multipliers have two real degrees of freedom, say β . Since the Painlevé equation II equation, Eq. (2.43), is the second order ODE system, These two parameters are the non-perturbative ambiguity of the system.

As is mentioned in Introduction, among these Stokes multipliers satisfying the algebraic relation (2.66), there is a special value which realizes the perturbative behavior (in $t \rightarrow \pm\infty$) of the matrix models argued from the physical point of views [42]. This special value is given as

$$\alpha = 0, \quad \beta = \pm 1, \quad (2.67)$$

and corresponds to the Hastings-McLeod solution in the Painlevé II equation [89]. From the mathematical point of view, this solution also has a good analytic behavior along the real isomonodromy parameter (cosmological constant) t [89, 100]. From this two-cut example, we generally expect that there is a special class of solutions of the Stokes multipliers which corresponds to the physical D-instanton chemical potentials. To generalize the solutions to the cases of arbitrary number of cuts, it is natural to ask the following question: *what is the physical requirements which specify the above multipliers?* This is also related to the issue cited by [61, 63]: *What is the boundary condition in continuum formulations which can fix the D-instanton chemical potentials in the matrix models?* Our procedure (discussed in Section 4 and Section 5) gives an answer to the question. In Section 4.2.1 and then in Section 5.1, we will see that our physical requirements correctly choose this particular parametrization Eq. (2.67) of the Stokes multipliers.

3 Stokes phenomena in the multi-cut cases

In this section, we develop general framework for Stokes phenomena in the general multi-cut critical points, and show explicitly *how the actual systems can be controlled*. Key information is provided by *profile of dominant exponents* (Theorem 3), and with this terminology we propose a systematic way to read the non-trivial Stokes multipliers (Theorem 4). Since the following discussions are *valid in general $k \times k$ ODE systems* of the following type:

$$\frac{d\Psi(t; \zeta)}{d\zeta} = (\Gamma^{-\gamma} \zeta^{r-1} + \cdots) \Psi(t; \zeta), \quad \text{g.c.d.}(k, \gamma) = 1, \quad (3.1)$$

we here develop the general framework without restricting to the \mathbb{Z}_k symmetry ($\gamma = r$). The restriction to the \mathbb{Z}_k symmetric cases only appear in Section 3.3.

3.1 Stokes lines and Stokes sectors

First we focus on the Stokes lines,

$$\text{SL}_{j,l} : \quad \text{Re}[(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)})\zeta^r] = 0, \quad (3.2)$$

and the resulting Stokes sectors (2.36). The leading coefficient of the exponents, $\varphi_{-r}^{(j)}$, which we consider here is given as¹³

$$\varphi_{-r}^{(j)} = \omega^{-\gamma(j-1)}. \quad (3.3)$$

Consequently, the conditions on the Stokes lines (in terms of angle, $\zeta = |\zeta|e^{i\theta}$) are expressed as

$$\operatorname{Re}[(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)})e^{ir\theta}] = 2 \sin(r\theta - \pi \frac{\gamma(j+l-2)}{k}) \sin(\pi \frac{\gamma(j-l)}{k}). \quad (3.4)$$

First of all, if there is a pair of (j, l) such that

$$\gamma(j-l) \in k\mathbb{Z}, \quad (3.5)$$

then the condition (2.31) does not satisfy. This means that the highest exponents degenerate $(\varphi_{-r}^{(j)} - \varphi_{-r}^{(l)})\zeta^r = 0$. In this case, we consider the next leading Stokes lines,

$$\operatorname{Re}[(\varphi_{-r+1}^{(j)} - \varphi_{-r+1}^{(l)})\zeta^{r-1}] = 0, \quad (3.6)$$

or more generally we consider the following Stokes lines:¹⁴

Definition 5 (General Stokes lines) *The general Stokes lines $\text{GSL}_{j,l}$ in this ODE system are defined for each pair of (j, l) as*

$$\text{GSL}_{j,l} \equiv \left\{ \zeta \in \mathbb{C}; \operatorname{Re}[\varphi^{(j)}(t; \zeta) - \varphi^{(l)}(t; \zeta)] = 0 \right\} = \bigcup_{n=0}^{2r-1} \text{GSL}_{j,l}^{(n)}, \quad (3.7)$$

which consists of $2r$ semi-infinite lines, $\text{GSL}_{j,l}^{(n)}$ ($n = 0, 1, \dots, 2r-1$). The set of lines, GSL , denotes a set of whole (general) Stokes lines, $\text{GSL} \equiv \bigcup_{j,l} \text{GSL}_{j,l}$.

The situations (3.5) are also interesting critical points in the multi-cut matrix models, however here for sake of simplicity, we concentrate on the following cases,

$$\text{g.c.d.}(k, \gamma) = 1, \quad (3.8)$$

because Eq. (3.5) becomes trivial in this case:

$$\gamma(j-l) \in k\mathbb{Z} \quad \Leftrightarrow \quad j-l \in k\mathbb{Z}. \quad (3.9)$$

Therefore Eq. (3.4) gives the angle $\theta_{j,l}^{(n)}$ for the Stokes lines $\text{SL}_{j,l}$ as

$$\theta = \theta_{j,l}^{(n)} = \frac{kn + \gamma(j+l-2)}{rk} \pi, \quad n \in \mathbb{Z}. \quad (3.10)$$

From this formula, one can read several basic information about the Stokes lines. An example of Stokes lines (3-cut (1, 1) case) is shown in Fig. 2-b. For later convenience, we introduce the following terminology:

¹³Note that the cases of our interest in the later sections are the \mathbb{Z}_k -symmetric critical points, and as one can see in Appendix B, the cases are given by $\gamma = r$. Also for future reference, we note that the fractional-superstring cases are given by $\gamma = r-2$.

¹⁴The physical interpretation of these general Stokes lines is the positions of eigenvalues in the matrix models. This viewpoint is also essential in this paper and discussed in Section 4.2.

Definition 6 (Segments) *Angular domains in between two Stokes lines which do not include any Stokes lines are called segments.*

In our present cases with a coprime (k, γ) , there are $2rk$ distinct segments δD_n ($n = 0, 1, \dots, 2rk - 1$) given as

$$\delta D_n \equiv D(n\delta\theta - \delta\theta, n\delta\theta), \quad \left(n = 0, 1, \dots, 2rk - 1; \delta\theta = \frac{\pi}{rk}\right), \quad (3.11)$$

which can fill the complex plane \mathbb{C} ,

$$\bigcup_{n=0}^{2rk-1} \overline{\delta D_n} = \mathbb{C}, \quad \delta D_m \cap \delta D_{m'} = \emptyset \quad (m \neq m'). \quad (3.12)$$

According to the definition of Stokes sectors, Eq. (2.36), we define the following most basic Stokes sectors, D_n :

Definition 7 (Fine Stokes sectors/matrices) *The following angular domains D_n*

$$D_n = e^{ni\delta\theta} D_0, \quad D_0 = D(-\delta\theta, k\delta\theta), \quad (n = 0, 1, \dots, 2rk - 1), \quad (3.13)$$

are Stokes sectors of a coprime (k, r) system with $k \geq 3$, which are referred to as fine Stokes sectors. The canonical solution of the fine Stokes sector D_n is denoted as $\tilde{\Psi}_n(t; \zeta)$ and the corresponding Stokes matrices S_n are given as

$$\tilde{\Psi}_{n+1}(t; \zeta) = \tilde{\Psi}_n(t; \zeta) S_n, \quad (3.14)$$

which is referred to as (fine) Stokes matrices.

Here we also define the other two kinds of Stokes sectors/matrices: First we define Stokes sectors/matrices which respect to the \mathbb{Z}_k symmetry of the multi-cut matrix models:

Definition 8 (Symmetric Stokes sectors/matrices) *The following subset of the fine Stokes sectors,*

$$D_{2nr}, \quad (n = 0, 1, \dots, k - 1), \quad (3.15)$$

are referred to as symmetric Stokes sectors,¹⁵ and the corresponding Stokes matrices $S_{2rn}^{(\text{sym})}$

$$S_{2rn}^{(\text{sym})} \equiv \tilde{\Psi}_{2rn}^{-1}(t; \zeta) \tilde{\Psi}_{2r(n+1)}(t; \zeta) = S_{2rn} \cdot S_{2rn+1} \cdots S_{2r(n+1)-1}. \quad (3.16)$$

are referred to as symmetric Stokes matrices.

Next we define the following economical Stokes sectors/matrices:

¹⁵Note that this definition is not enough for the $k = 3, r = 2$ case. In these cases, we employ a modified version of the Stokes sectors, for example, D_{nr} .

Definition 9 (Coarse Stokes sectors/matrices) *The following subset of the fine Stokes sectors,*

$$D_{nk}, \quad (n = 0, 1, \dots, 2r - 1), \quad (3.17)$$

are referred to as coarse Stokes sectors, and the corresponding Stokes matrices $S_{nk}^{(c)}$ are written as

$$S_{nk}^{(\text{coa})} \equiv \tilde{\Psi}_{nk}^{-1}(t; \zeta) \tilde{\Psi}_{(n+1)k}(t; \zeta) = S_{nk} \cdot S_{nk+1} \cdots S_{(n+1)k-1}. \quad (3.18)$$

are referred to as coarse Stokes matrices.

Coarse Stokes sectors are most often used in the literature. However, in the following discussions, one will see that the fine Stokes matrices are more convenient for our calculations.

3.2 Stokes multipliers from the profile of dominant exponents

In principle, one can use Theorem 2 to read the non-trivial (or non-zero) Stokes multipliers in each specific case. That is the problem of finding which components can take non-zero value in Stokes matrices. However, practically in general, it is tedious to use this standard way to read the non-trivial multipliers, especially in the higher $k \times k$ system with higher Poincaré index r . The purpose of this section is therefore to point out an interesting connection between the non-trivial Stokes multipliers and *profile of dominant exponents* which we develop in this subsection (Theorem 3 and 4). An important thing in this procedure is that these results make it easy to put data of the Stokes multipliers in computer, for example, in Mathematica program.

Since there is no Stokes line in the segments defined in Eq. (3.11), one can define the following ordered set J_l of indices $j_{l,i}$:

$$J_l = [j_{l,1} \mid j_{l,2} \mid \cdots \mid j_{l,k}] \in \mathbb{N}^k, \quad (3.19)$$

which describes the profile of dominant exponents in the segment D_l

$$\text{Re}[\varphi_{-r}^{(j_{l,1})} \zeta^r] < \text{Re}[\varphi_{-r}^{(j_{l,2})} \zeta^r] < \cdots < \text{Re}[\varphi_{-r}^{(j_{l,k})} \zeta^r], \quad \zeta \in \delta D_l. \quad (3.20)$$

This sequence of numbers, $\mathcal{J} = \{J_l\}_{l=0}^{2rk}$, is referred to as *profile of dominant exponents*. Here we express the profile \mathcal{J} as follows:

$$\mathcal{J} = \left[\begin{array}{c|c|c|c} j_{2rk-1,1} & j_{2rk-1,2} & \cdots & j_{2rk-1,k} \\ \hline \vdots & \vdots & & \vdots \\ \hline j_{1,1} & j_{1,2} & \cdots & j_{1,k} \\ \hline j_{0,1} & j_{0,2} & \cdots & j_{0,k} \end{array} \right] \quad (3.21)$$

Note that the ordering of indices in the vertical direction is different from the usual matrix, and that elements are periodic in the index l , $J_l = J_{l+2rk}$. An example (3-cut (1, 1) critical point) and the relation to the ζ plane are shown in Fig. 5.

The non-trivial problem for the profile is then how to fill the numbers in the profiles. We found the following simple answer:

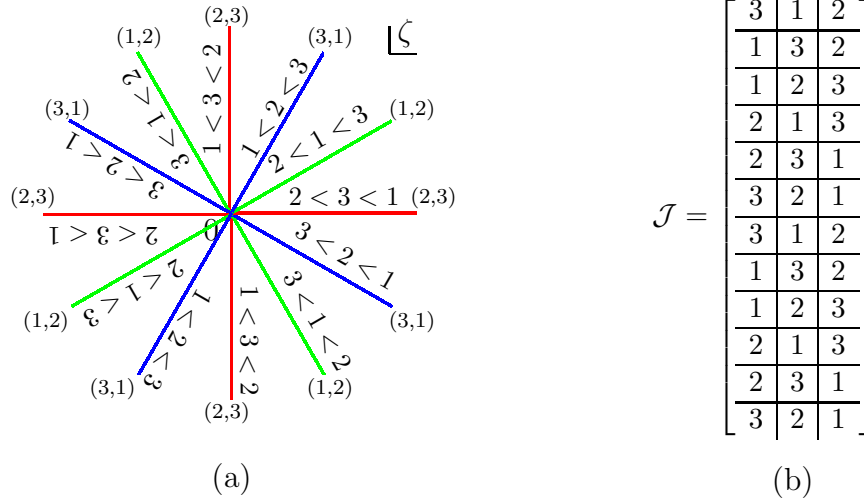


Figure 5: The two expressions for the profile of dominant exponents in the 3-cut (1,1) critical point. $\text{Re}[\varphi_{-2}^{(j_{l,1})}] < \text{Re}[\varphi_{-2}^{(j_{l,2})}] < \text{Re}[\varphi_{-2}^{(j_{l,3})}]$ is expressed as $j_{l,1} < j_{l,2} < j_{l,3}$. a) The profile in the ζ plane. b) The profile \mathcal{J} in the table. In the same way, the dominance is expressed as $[j_{l,1}|j_{l,2}|j_{l,3}]$

Theorem 3 (General components) *The general components $j_{l,n}$ of the profile \mathcal{J} with $\text{g.c.d.}(k, \gamma) = 1$ are given as¹⁶*

$$j_{l,n} \equiv 1 + \left(\left\lfloor \frac{l}{2} \right\rfloor + (-1)^{k+l+n} \left\lfloor \frac{k-n+1}{2} \right\rfloor \right) m_1, \quad \text{mod } k, \quad (3.22)$$

where m_1 is obtained by the Euclidean algorithm of $kn_1 + \gamma m_1 = 1$.

Some comments on this formula are in order:

- In any segment profile $J_l = [j_{l,1} | j_{l,2} | \cdots | j_{l,k}]$, a pair of indices (i, j) which change their relative dominance at angle $\theta = l\delta\theta$ are always next to each other, and they satisfy the following sum rule:¹⁷

$$i + j - 2 \equiv m_l (\equiv lm_1) \quad \text{mod } k, \quad (3.24)$$

with an integer m_l which is obtained by the Euclidean algorithm of $kn_l + \gamma m_l = l$. Therefore in particular, we represent these pairs as $(i|j)$ in the profile (See Eqs. (3.26)).

- Theorem 3 can be recursively shown by using the sum rules Eq. (3.24), and its initial conditions $j_{0,k} = j_{1,k} = 1 = j_{k,1} = j_{k+1,1}$ which can be easily checked.

¹⁶In this paper, we use the floor-function notation for the gauss symbol, $\lfloor a \rfloor$, which means the largest integer less than or equal to a .

¹⁷Note that the condition for $\theta_{i,j}^{(n)} = l\delta\theta$ is given as

$$\frac{kn + \gamma(i + j - 2)}{rk} \pi = \frac{l}{rk} \pi \quad \Leftrightarrow \quad kn + \gamma(i + j - 2) = l. \quad (3.23)$$

This means that, for a given l , find a pair (i, j) such that there exists an integer n .

- The trajectories of the indices, for instance a and b , are given as follows:

$$\begin{array}{cc}
\underline{k \text{ is odd}} & \underline{k \text{ is even}}
\end{array}$$

$$\mathcal{J} = \begin{bmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & a & & \cdots & b & & \\ a & & & \cdots & & b & \\ a & & & \cdots & & & b \\ & a & & \cdots & & & b \\ & & a & \cdots & & b & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ & a & & \cdots & & b & \\ a & & & \cdots & & & b \\ a & & & \cdots & & & b \\ & a & & \cdots & & b & \\ & & a & \cdots & b & & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \end{bmatrix} \quad (3.25)$$

- The above formula is not applicable in the case of $\text{g.c.d}(k, \gamma) \neq 1$, since some exponents degenerate, but one example of this kind is shown in Appendix F.

Here also two examples of the profiles $\mathcal{J}_{k,r}$ are shown for the case of $(k, r) = (3, 2)$ and $(5, 2)$ with $\gamma = r$ (\mathbb{Z}_k symmetry condition):

$$\mathcal{J}_{3,2} = \begin{bmatrix} 3 & (1 & 2) \\ (1 & 3) & 2 \\ 1 & (2 & 3) \\ (2 & 1) & 3 \\ 2 & (3 & 1) \\ (3 & 2) & 1 \\ 3 & (1 & 2) \\ (1 & 3) & 2 \\ 1 & (2 & 3) \\ (2 & 1) & 3 \\ 2 & (3 & 1) \\ (3 & 2) & 1 \end{bmatrix}, \quad \mathcal{J}_{5,2} = \begin{bmatrix} 2 & (4 & 5) & (1 & 3) \\ (4 & 2) & (1 & 5) & 3 \\ 4 & (1 & 2) & (3 & 5) \\ (1 & 4) & (3 & 2) & 5 \\ 1 & (3 & 4) & (5 & 2) \\ (3 & 1) & (5 & 4) & 2 \\ 3 & (5 & 1) & (2 & 4) \\ (5 & 3) & (2 & 1) & 4 \\ 5 & (2 & 3) & (4 & 1) \\ (2 & 5) & (4 & 3) & 1 \\ 2 & (4 & 5) & (1 & 3) \\ (4 & 2) & (1 & 5) & 3 \\ 4 & (1 & 2) & (3 & 5) \\ (1 & 4) & (3 & 2) & 5 \\ 1 & (3 & 4) & (5 & 2) \\ (3 & 1) & (5 & 4) & 2 \\ 3 & (5 & 1) & (2 & 4) \\ (5 & 3) & (2 & 1) & 4 \\ 5 & (2 & 3) & (4 & 1) \\ (2 & 5) & (4 & 3) & 1 \end{bmatrix}. \quad (3.26)$$

One can observe that there is a $2k$ periodicity, $j_{l+2k,n} = j_{l,n}$, or more precisely, a reflection by step k , $j_{l,n} = j_{l+k,k-n+1}$.

Next we demonstrate how to read the non-trivial Stokes multipliers in some examples, and see the general rule. In the case of $(r, k; \gamma) = (2, 5; 2)$ and its symmetric Stokes matrix $S_0^{(\text{sym})}$, one first sees the dominance profile in the domain $D_0 \cap D_4$,

$$D_0 \cap D_4 \supset \left[\begin{array}{c|c|c|c|c} 1 & 3 & 4 & 5 & 2 \\ \hline 3 & 1 & 5 & 4 & 2 \end{array} \right], \quad \begin{array}{l} \leftarrow J_5 \\ \leftarrow J_4 \end{array} \quad (3.27)$$

and reads the ordering of magnitude:

$$(2) > (5), (4), (3), (1), \quad (5) > (3), (1), \quad (4) > (3), (1). \quad (3.28)$$

This results in the following symmetric Stokes multipliers:

$$S_0^{(\text{sym})} = \begin{pmatrix} 1 & s_{0,1,2}^{(\text{sym})} & 0 & s_{0,1,4}^{(\text{sym})} & s_{0,1,5}^{(\text{sym})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\text{sym})} & 1 & s_{0,3,4}^{(\text{sym})} & s_{0,3,5}^{(\text{sym})} \\ 0 & s_{0,4,2}^{(\text{sym})} & 0 & 1 & 0 \\ 0 & s_{0,5,2}^{(\text{sym})} & 0 & 0 & 1 \end{pmatrix}. \quad (3.29)$$

In the same way, for the calculation of the fine Stokes matrix S_0 , one first sees the dominance profile in the domain $D_0 \cap D_1$,

$$D_0 \cap D_1 \supset \left[\begin{array}{c|c|c|c|c} 1 & \mathbf{3} & 4 & \underline{5} & \underline{2} \\ \hline \mathbf{3} & 1 & \underline{5} & 4 & \underline{2} \\ \hline \mathbf{3} & \underline{5} & 1 & \underline{2} & 4 \\ \hline \underline{5} & \mathbf{3} & \underline{2} & 1 & 4 \\ \hline \underline{5} & \underline{2} & \mathbf{3} & 4 & 1 \end{array} \right], \quad (3.30)$$

$\leftarrow J_5$

$\leftarrow J_1$

and reads the ordering of magnitude:

$$(4) > (3), \quad (2) > (5). \quad (3.31)$$

This results in the Stokes multipliers:

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & s_{0,3,4} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix}. \quad (3.32)$$

These are the standard way of reading the multipliers.¹⁸

However, one may notice that there is a relation between indices of non-zero Stokes multipliers $s_{0,i,j}$ in the Stokes matrix S_0 and the dominance-changing pairs $(j|i)$ in the profile J_0 :

$$s_{0,3,4}, \quad s_{0,5,2} \quad \leftrightarrow \quad (2|5), \quad (4|3) \quad \in \quad J_0 = [(2 | 5) | (4 | 3) | 1]. \quad (3.33)$$

We claim that this observation is generally true:

Theorem 4 (Stokes multipliers from the profiles) *The non-zero Stokes multipliers in the fine Stokes matrix S_l have a correspondence with dominance-changing pairs $(j|i)$ in the profile J_l as follows:*

$$s_{l,i,j} \ (i \neq j) \text{ can take non-zero value} \quad \Leftrightarrow \quad (j|i) \in J_l. \quad (3.34)$$

Note that the orderings of indices $(i|j)$ and $s_{l,j,i}$ are opposite $i \leftrightarrow j$.

¹⁸ From this procedure, one may notice that the simplest choice is the coarse Stokes sectors $S_{nk}^{(\text{coa})}$, because intersections have the definite order of magnitude: $D_0 \cap D_5 \supset [1 | 3 | 4 | 5 | 2]$ and the number of Stokes matrices is the smallest. This is the main reason why the coarse Stokes sectors are often used in literature. However, we will see that the coarse Stokes multipliers are not suitable for general formula of higher k and r at least in the \mathbb{Z}_k symmetric critical points.

A proof is easy if one notices that intersections of fine Stokes sectors $D_n \cap D_{n+1}$ are always a half of the period of Stokes line formula Eq. (3.4). The other Stokes matrices, say $S_n^{(\text{sym})}$ and $S_n^{(\text{coa})}$, are written as a product of the fine Stokes matrices S_n (as in (3.16) and (3.18)). For instance,

$$\begin{aligned}
S_0^{(\text{sym})} &= \begin{pmatrix} 1 & s_{0,1,2}^{(\text{sym})} & 0 & s_{0,1,4}^{(\text{sym})} & s_{0,1,5}^{(\text{sym})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\text{sym})} & 1 & s_{0,3,4}^{(\text{sym})} & s_{0,3,5}^{(\text{sym})} \\ 0 & s_{0,4,2}^{(\text{sym})} & 0 & 1 & 0 \\ 0 & s_{0,5,2}^{(\text{sym})} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & s_{2,1,2} + s_{1,1,4}s_{3,4,2} & 0 & s_{1,1,4} & s_{3,1,5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{1,3,2} + s_{0,3,4}s_{3,4,2} & 1 & s_{0,3,4} & s_{2,3,5} \\ 0 & s_{3,4,2} & 0 & 1 & 0 \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix}, \\
S_0^{(\text{coa})} &= \begin{pmatrix} 1 & s_{0,1,2}^{(\text{coa})} & s_{0,1,3}^{(\text{coa})} & s_{0,1,4}^{(\text{coa})} & s_{0,1,5}^{(\text{coa})} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{0,3,2}^{(\text{coa})} & 1 & s_{0,3,4}^{(\text{coa})} & s_{0,3,5}^{(\text{coa})} \\ 0 & s_{0,4,2}^{(\text{coa})} & 0 & 1 & s_{0,4,5}^{(\text{coa})} \\ 0 & s_{0,5,2}^{(\text{coa})} & 0 & 0 & 1 \end{pmatrix} = \\
&= \begin{pmatrix} 1 & s_{2,1,2} + s_{1,1,4}s_{3,4,2} & s_{4,1,3} & s_{1,1,4} & s_{3,1,5} + s_{1,1,4}s_{4,4,5} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & s_{1,3,2} + s_{0,3,4}s_{3,4,2} & 1 & s_{0,3,4} & s_{2,3,5} + s_{0,3,4}s_{4,4,5} \\ 0 & s_{3,4,2} & 0 & 1 & s_{4,4,5} \\ 0 & s_{0,5,2} & 0 & 0 & 1 \end{pmatrix} \quad (3.35)
\end{aligned}$$

As one can see from these special examples, the Stokes multipliers are always related as

$$s_{0,i,j}^{(\text{xxx})} = s_{*,i,j} + \dots, \quad (3.36)$$

and one can then show that the number of independent Stokes multipliers in each Stokes matrix $S_0^{(\text{sym})}$ and $S_0^{(\text{coa})}$ is the same and is supplied by the fine Stokes matrices. Consequently, the same statement also holds for these different kinds of Stokes multipliers: For example,

$$\begin{aligned}
&s_{2r(l-1),i,j}^{(\text{sym})} \ (i \neq j) \text{ can take non-zero value} \\
&\Leftrightarrow \ (j|i) \in J_n, \quad n = 2r(l-1), 2r(l-1) + 1, \dots, 2rl - 1. \quad (3.37)
\end{aligned}$$

3.3 Three basic constraints on the Stokes matrices

Finally we show the three basic constraints on the Stokes multipliers, which result from the detail analysis of (the \mathbb{Z}_k -symmetric) critical points in the multi-cut two-matrix models [46] and also which provide a natural extension of the two-cut cases (See Section 2.3). We should note that the conditions from the matrix models are given in the Γ -basis (or the matrix-model basis) $\Psi(t; \zeta)$ and the Stokes matrices are defined in the Ω -basis (the diagonal basis) $\tilde{\Psi}(t; \zeta)$, and they are related by a unitary transformation (See Eq. (2.22) and Eq. (2.23)).

\mathbb{Z}_k symmetry condition This condition is from the Z_k symmetry in the multi-cut two-matrix models [46] and generally expressed as¹⁹

$$\omega^{-1} \mathcal{Q}(t; \omega^{-1} \zeta) = \Omega^{-1} \mathcal{Q}(t; \zeta) \Omega, \quad \mathcal{P}(t; \omega^{-1} \zeta) = \Omega^{-1} \mathcal{P}(t; \zeta) \Omega, \quad (3.38)$$

¹⁹This is a direct consequence of Eqs. (2.6).

with $\Omega^{-1}E_{i,i+1}\Omega = \omega E_{i,i+1}$. The constraint on the Stokes matrices are then obtained as

$$S_{n+2r} = \Gamma^{-1}S_n\Gamma, \quad (n = 0, 1, \dots, 2rk - 1) \quad (3.39)$$

for the fine Stokes matrices S_n . A note for the derivation is following:

1. Because of the condition Eq. (3.38), the canonical solution $\Psi_n(t; \zeta)$ for a Stokes sector D_n satisfies

$$g_{\text{str}} \frac{\partial [\Omega \Psi_n(t; \omega^{-1}\zeta) \Omega^{-1}]}{\partial \zeta} = \mathcal{Q}(t; \zeta) [\Omega \Psi_n(t; \omega^{-1}\zeta) \Omega^{-1}], \quad (3.40)$$

and therefore one obtains

$$\Psi_{n+2r}(t; \zeta) = [\Omega \Psi_n(t; \omega^{-1}\zeta) \Omega^{-1}] \underset{\text{asym}}{\simeq} \Psi_{\text{asym}}(t; \zeta), \quad \zeta \rightarrow \infty \in D_{n+2r} = D_n. \quad (3.41)$$

2. By translating this relation into the Ω -basis (diagonal basis),

$$\Psi_{n+2r}(t; \zeta) = \Omega \Psi_n(t; \omega^{-1}\zeta) \Omega^{-1} \Leftrightarrow \tilde{\Psi}_{n+2r}(t; \zeta) = \Gamma^{-1} \tilde{\Psi}_n(t; \omega^{-1}\zeta) \Gamma, \quad (3.42)$$

with $U^{-1}\Omega U = \Gamma^{-1}$, one obtains the relation of the Stokes matrices:

$$S_{n+2r} = \tilde{\Psi}_{n+2r}^{-1}(t; \zeta) \tilde{\Psi}_{n+2r+1}(t; \zeta) = \Gamma^{-1} \tilde{\Psi}_n^{-1}(t; \zeta) \tilde{\Psi}_{n+1}(t; \zeta) \Gamma = \Gamma^{-1} S_n \Gamma. \quad (3.43)$$

This condition means that only the first $2r$ Stokes matrices S_n ($n = 0, 1, \dots, 2r - 1$) are independent. Therefore, we use the first $2r$ dominance profiles to identify the non-trivial Stokes multipliers:

$$\mathcal{J}_{k,r}^{(\text{sym})} \equiv \begin{bmatrix} \frac{J_{2r-1}}{J_1} \\ \vdots \\ \frac{J_1}{J_0} \end{bmatrix} \Leftrightarrow S_n \quad (n = 0, 1, \dots, 2r - 1). \quad (3.44)$$

Here we show an examples of $k = 5, r = \gamma = 2$:

$$\mathcal{J}_{5,2}^{(\text{sym})} = \begin{bmatrix} \begin{array}{c|c|c|c|c} 3 & (5) & 1 & (2) & 4 \\ \hline (5) & 3 & (2) & 1 & 4 \\ \hline 5 & (2) & 3 & (4) & 1 \\ \hline (2) & 5 & (4) & 3 & 1 \end{array} \end{bmatrix} : \begin{matrix} J_3 \\ J_2 \\ J_1 \\ J_0 \end{matrix}. \quad (3.45)$$

Hermiticity condition This condition is from the hermiticity of the multi-cut matrix models [46] and generally expressed as

$$\mathcal{Q}^*(t; \zeta^*) = \mathcal{Q}(t; \zeta^*). \quad (3.46)$$

The constraints on the Stokes matrices are then obtained as

$$S_n^* = \Delta \Gamma S_{(2r-1)k-n}^{-1} \Gamma^{-1} \Delta, \quad \Delta_{i,j} = \delta_{i,k-j+1}, \quad (n = 0, 1, \dots, 2rk - 1) \quad (3.47)$$

for the fine Stokes matrices S_n . A note for the derivation is following:

1. Because of the condition Eq. (3.46), the canonical solution $\Psi_n(t; \zeta)$ for a Stokes sector D_n satisfies

$$g_{\text{str}} \frac{\partial \Psi_n^*(t; \zeta)}{\partial \zeta} = \mathcal{Q}(t; \zeta) \Psi_n^*(t; \zeta), \quad (3.48)$$

and therefore one obtains

$$\Psi_{(2r-1)k+1-n}(t; \zeta) = \Psi_n^*(t; \zeta) \underset{\text{asym}}{\simeq} \Psi_{\text{asym}}(t; \zeta), \quad \zeta \in D_{(2r-1)k+1-n} = D_n^*. \quad (3.49)$$

2. By translating this relation into the Ω -basis (diagonal basis),

$$\begin{aligned} \Psi_n^*(t; \zeta) = \Psi_{(2r-1)k+1-n}(t; \zeta) &\Leftrightarrow \tilde{\Psi}_n^* = U^2 \tilde{\Psi}_{(2r-1)k+1-n}(t; \zeta) U^{-2} \\ &= \Delta \Gamma \tilde{\Psi}_{(2r-1)k+1-n}(t; \zeta) \Gamma^{-1} \Delta, \end{aligned} \quad (3.50)$$

with $U^* = U^{-1}$ and $U^2 = \Delta \Gamma$, one obtains the relation of the Stokes matrices

$$\begin{aligned} S_n^* &= [\tilde{\Psi}_n^{-1}(t; \zeta) \tilde{\Psi}_{n+1}(t; \zeta)]^* = \Delta \Gamma [\tilde{\Psi}_{(2r-1)k+1-n}^{-1}(t; \zeta) \tilde{\Psi}_{(2r-1)k-n}(t; \zeta)] \Gamma^{-1} \Delta \\ &= \Delta \Gamma [\tilde{\Psi}_{(2r-1)k-n}^{-1}(t; \zeta) \tilde{\Psi}_{(2r-1)k+1-n}(t; \zeta)]^{-1} \Gamma^{-1} \Delta \\ &= \Delta \Gamma S_{(2r-1)k-n}^{-1} \Gamma^{-1} \Delta. \end{aligned} \quad (3.51)$$

Monodromy free condition If the formal expansion satisfies $\varphi_0 = 0$ (discussed in Appendix B), then the canonical solutions are the single-valued functions:

$$\tilde{\Psi}_n(t; \zeta) = \tilde{\Psi}_n(t; e^{2\pi i} \zeta) = \tilde{\Psi}_{n+2kr}(t; \zeta), \quad (3.52)$$

therefore the Stokes matrices satisfy

$$\begin{aligned} S_0 \cdot S_1 \cdots S_{2rk-1} &= S_0^{(\text{coa})} \cdot S_k^{(\text{coa})} \cdots S_{k(2r-1)}^{(\text{coa})} \\ &= S_0^{(\text{sym})} \cdot S_{2r}^{(\text{sym})} \cdots S_{2r(k-1)}^{(\text{sym})} = I_k. \end{aligned} \quad (3.53)$$

Note that, with the \mathbb{Z}_k -symmetry constraints, $2rk$ Stokes matrices are reduced to fundamental $2r$ Stokes matrices, $\{S_n\}_{n=0}^{2r-1}$, and also that the monodromy free condition is written as

$$(S_0^{(\text{sym})} \Gamma^{-1})^k = I_k. \quad (3.54)$$

4 The multi-cut boundary condition and solutions

In the previous section, we developed the general framework of Stokes phenomena in the ODE systems which appear in the multi-cut matrix models. Mathematically, general solutions $\Psi(t; \zeta)$ for these isomonodromy systems (or equivalently for the corresponding Douglas (string) equations) are parametrized by the Stokes multipliers with three basic constraints discussed in Section 3.3. As is mentioned in Introduction, however, not all the solutions to these constraints can realize the critical points in the multi-cut matrix models. This consideration requires *additional physical constraints* on the Stokes multipliers. In this section, the first *physical constraint* is proposed, which we refer to as *multi-cut boundary conditions*. The second physical condition is proposed in Section 5.

4.1 Two different viewpoints about spectral curves

Before we discuss the detail of the multi-cut boundary conditions, we first recall the set up of the multi-cut two-matrix models and the relationship between the Baker-Akhiezer function system (i.e. the ODE system) and the resolvent operator which defines the spectral curves.

The definition of the multi-cut two-matrix models is given by the following matrix integral:

$$Z = \int_{\mathcal{C}_N^{(k)} \times \mathcal{C}_N^{(k)}} dX dY e^{-N \text{tr}[V_1(X) + V_2(Y) - XY]}, \quad (4.1)$$

with the matrix contour $\mathcal{C}_N^{(k)}$ of the following $N \times N$ k -cut normal matrix,

$$\mathcal{C}_N^{(k)} \equiv \left\{ X = U \text{diag}(x_1, x_2, \dots, x_N) U^\dagger; U \in U(N), x_j \in \bigcup_{n=0}^{k-1} e^{2\pi i \frac{n}{k}} \mathbb{R} \right\}. \quad (4.2)$$

The system of two-matrix models has the corresponding orthonormal polynomial system [104]:

$$\alpha_n(x) = \frac{1}{\sqrt{h_n}} (x^n + \dots), \quad \beta_n(y) = \frac{1}{\sqrt{h_n}} (y^n + \dots), \quad (4.3)$$

with

$$\delta_{n,m} = \int_{\mathcal{C}^{(k)} \times \mathcal{C}^{(k)}} dx dy e^{-N[V_1(x) + V_2(y) - xy]} \alpha_n(x) \beta_m(y). \quad (4.4)$$

Here the contour $\mathcal{C}^{(k)}$ is given as

$$\mathcal{C}^{(k)} = \left\{ x \in \bigcup_{n=0}^{k-1} e^{2\pi i \frac{n}{k}} \mathbb{R} \right\}, \quad (4.5)$$

an example of which is shown in Fig. 6.

The Baker-Akhiezer systems (or the ODE systems) appear as the double scaling limit of the orthonormal polynomials $\alpha_n(x)$ (or their dual polynomials $\beta_n(y)$), which is given as follows:

$$\alpha_n(x) = a^{-\hat{p}/2} \Psi_{\text{orth}}(\zeta; t), \quad (4.6)$$

with the following scaling relations of $a \rightarrow 0$:

$$\begin{aligned} x = \omega^{-1/2} a^{\hat{p}/2} \zeta &\rightarrow 0, & \frac{n}{N} &= \exp(-ta^{\frac{\hat{p}+\hat{q}-1}{2}}) \rightarrow 1, \\ N^{-1} = g_{\text{str}} a^{\frac{\hat{p}+\hat{q}}{2}} &\rightarrow 0, & \partial_n &= -a^{1/2} g_{\text{str}} \partial_t \equiv -a^{1/2} \partial \rightarrow 0. \end{aligned} \quad (4.7)$$

The continuous function $\Psi_{\text{orth}}(t; \zeta)$ is the scaling function of the orthonormal polynomials and satisfies the differential equations (4.8) and (4.9):

$$\zeta \Psi_{\text{orth}}(t; \zeta) = \mathbf{P}(t; \partial) \Psi_{\text{orth}}(t; \zeta), \quad (4.8)$$

$$g_{\text{str}} \frac{\partial}{\partial \zeta} \Psi_{\text{orth}}(t; \zeta) = \mathbf{Q}(t; \partial) \Psi_{\text{orth}}(t; \zeta). \quad (4.9)$$

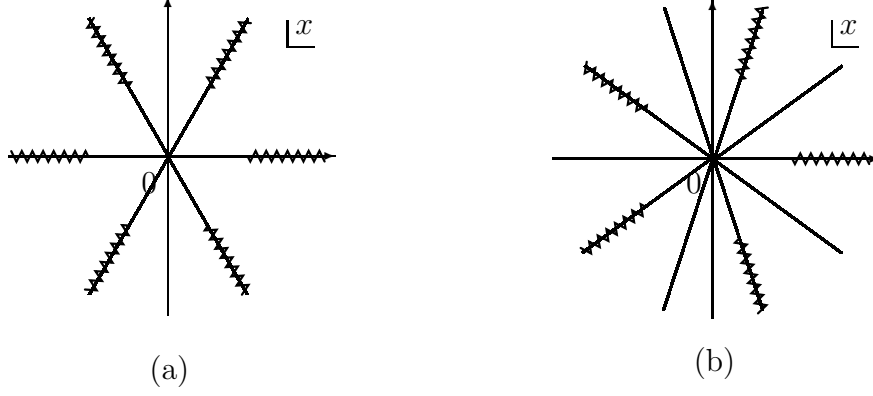


Figure 6: Examples of contours $\mathcal{C}^{(k)}$. (a) is 6-cut contour $\mathcal{C}^{(6)}$ and (b) is the 5-cut contour $\mathcal{C}^{(5)}$ which is equal to the 10-cut contour $\mathcal{C}^{(10)}$. For reference, the position of cuts (zig-zag lines) around $\zeta \rightarrow \infty$ is also denoted.

This means that the orthonormal polynomial system is *one of the solutions* to the differential equations (4.8) and (4.9), and eventually the ODE systems (2.13) and (2.14). Consequently, the scaling function $\Psi_{\text{orth}}(t; \zeta)$ is given by the canonical solutions $\Psi_n(t; \zeta)$ with some proper vector $X^{(n)} = {}^t(x_1^{(n)}, x_2^{(n)}, \dots, x_k^{(n)})$ as

$$\Psi_{\text{orth}}(t; \zeta) = \tilde{\Psi}_n(t; \zeta) X^{(n)}, \quad (n = 0, 1, \dots). \quad (4.10)$$

By taking into account the Stokes phenomena (3.14), these vectors $X^{(n)}$ of various Stokes sectors D_n are related as follows:

$$X^{(n)} = S_n X^{(n+1)}, \quad X^{(n+2rk)} = X^{(n)}. \quad (4.11)$$

Note that the scaled orthonormal polynomials $\Psi_{\text{orth}}(t; \zeta)$ are entire functions in $\zeta \in \mathbb{C}$ because the original orthonormal polynomials are also entire functions.

On the other hand, another important approach to solving the multi-cut matrix models is the semi-classical approach with the resolvent operator $\mathcal{R}(x)$ of the matrix models,

$$\mathcal{R}(x) = \left\langle \frac{1}{N} \text{tr} \frac{1}{x - X} \right\rangle = \int_{\mathcal{C}^{(k)}} dz \frac{\rho(z)}{x - z}, \quad (4.12)$$

where $\rho(z)$ is the density function of eigenvalues of the matrix X . An important fact about the resolvent is that this operator is a single valued function in $x \in \mathbb{C}$ with a finite N and the cuts appearing in the large N limit are along the matrix-model contour $\mathcal{C}^{(k)}$ on the x space (as shown in Fig. 6). These special cuts are called *physical cuts*. Interestingly, this resolvent operator is also related to the orthonormal polynomial solution $\Psi_{\text{orth}}(t; \zeta)$ in the following way [20]:²⁰

$$\Psi_{\text{orth}}(t; \zeta) \sim \langle \det(x - X) \rangle \sim \exp \left[N \int^x dx' \mathcal{R}(x') \right], \quad (4.13)$$

²⁰For the precise relations, see Appendix A in [46], for example.

with the scaling relation, $x = \omega^{-1/2} a^{\hat{p}/2} \zeta$, of Eq. (4.7). This relation also indicates²¹

$$\mathcal{R}(x) \sim \lim_{g_{\text{str}} \rightarrow 0} g_{\text{str}} \frac{\partial}{\partial \zeta} \ln \Psi_{\text{orth}}(t; \zeta). \quad (4.14)$$

Therefore, the two different observables, the semi-classical resolvent $\mathcal{R}(x)$ Eq. (4.12) and the semi-classical orthonormal polynomials $\Psi_{\text{orth}}(t; \zeta)$ Eq. (4.14), provide two different viewpoints of spectral curves.

4.2 The multi-cut boundary conditions

As we carefully see the above two viewpoints of the spectral curve, one can notice that the realization of *the position of physical cuts* is not straightforward from the Baker-Akhiezer (or ODE) approach. The discontinuities of the scaling orthonormal polynomial of Eq. (4.10) around $\zeta \rightarrow \infty$ are *Stokes lines* and generally not distributed in the proper way expected in the semi-classical resolvent operator Eq. (4.12). This eventually means that not all the solutions to the ODE system Eq. (4.8) and Eq. (4.9) (therefore equivalently string equations) correspond to critical points of the multi-cut matrix models. The difference between these two viewpoints provides additional *physical constraints* not only on the vectors $X^{(n)}$ but also on the Stokes multipliers $s_{l,i,j}$ which are identified as integration constants of the string equations.

Next we formulate this physical constraint in the following way. Note that we here only care the leading behavior of $\zeta \rightarrow \infty$ for the Stokes lines.

Definition 10 (Multi-cut boundary condition) *The Baker-Akhiezer (or ODE) systems Eq. (4.8) and Eq. (4.9) are said to satisfy the multi-cut boundary condition, if there exists a special solution $\Psi_{\text{orth}}(t; \zeta)$ which satisfies the following condition:*

- *Stokes lines of the solution $\Psi_{\text{orth}}(t; \zeta)$ around $\zeta \rightarrow \infty$ only exist along some special k angles $\zeta \rightarrow \infty \times e^{i\chi_n}$:*

$$\chi_n \equiv \chi_0 + \frac{2\pi n}{k} \quad (n = 0, 1, 2, \dots, k-1), \quad (4.15)$$

with a proper χ_0 corresponding to each critical point.

- *Therefore, there exist an ordered set of k indices, (a_1, a_2, \dots, a_k) , and a set of k non-zero vectors, (v_1, v_2, \dots, v_k) , such that the asymptotic expansions of the solution $\Psi_{\text{orth}}(t; \zeta)$ in the angular domain $D(\chi_n, \chi_{n+1})$ are given as*

$$\Psi_{\text{orth}}(t; \zeta) \underset{\text{asym}}{\simeq} v_n e^{\varphi^{(a_n)}(t; \zeta)} + \dots, \quad \zeta \rightarrow \infty \in D(\chi_n, \chi_{n+1}), \quad (4.16)$$

and the expansions along the Stokes lines are given as the superposition:

$$\Psi_{\text{orth}}(t; \zeta) \underset{\text{asym}}{\simeq} v_n e^{\varphi^{(a_n)}(t; \zeta)} + v_{n+1} e^{\varphi^{(a_{n+1})}(t; \zeta)} \dots, \quad \zeta \rightarrow \infty \times e^{i\chi_{n+1}}. \quad (4.17)$$

²¹ Although $\Psi_{\text{orth}}(t; \zeta)$ is a vector valued function, the behaviors of exponents are the same among the vector components. Therefore, it is understood by taking one particular element of the function $\Psi_{\text{orth}}(t; \zeta)$.

Here appears a special angle χ_0 which is determined by the critical points of the matrix models and is given as follows:

$$\chi_0 = \begin{cases} \frac{\pi}{k} & : \mathbb{Z}_k\text{-symmetric cases, and } \omega^{1/2}\text{-rotated FSST cases} \\ 0 & : \text{Real-potential FSST cases} \end{cases}. \quad (4.18)$$

This angle χ_0 comes from the scaling relation Eq. (4.7), for detail discussion of which we should refer to [46]. An example of the boundary condition in the ζ plane is shown in Fig. 7.

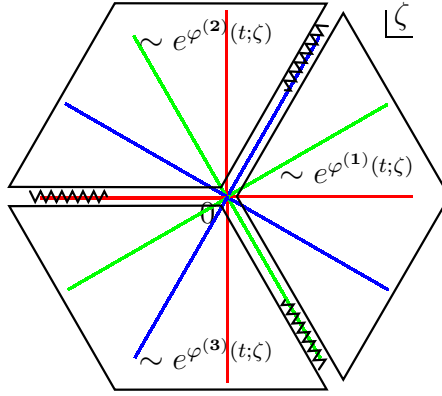


Figure 7: The multi-cut boundary condition in the 3-cut (1,1) critical point. Although the general solutions to the Baker-Akhiezer function system can generally have “twelve cuts”, there are only three cuts in the scaling orthonormal-polynomial solution $\Psi_{\text{orth}}(t; \zeta)$.

Some comments are in order:

- As in the definition of Stokes lines Eq. (2.35), we here only used the leading contributions of the exponents in $\zeta \rightarrow \infty$. If we also take $t \rightarrow \infty$, or equivalently if we just take $g_{\text{str}} \rightarrow 0$, on the other hand, we naturally encounter the general Stokes lines of Definition 5. Therefore, we interpret the general Stokes lines of the scaling orthonormal-polynomial solution as *the non-perturbative definition of physical cuts*. In particular, this definition guarantees *real eigenvalue-density function* $\rho(\lambda)d\lambda$ along the physical cuts:

$$\text{Re}[\pi i \rho(\lambda) d\lambda] \equiv \text{Re}[d\varphi^{(j)}(t; \zeta) - d\varphi^{(l)}(t; \zeta)] = 0, \quad (4.19)$$

where $\zeta = \zeta(\lambda)$ is a local map from $\lambda \in \mathbb{R}$ to the generalized Stokes line $\text{GSL}_{i,j} \subset \mathbb{C}$. Note that this definition naturally justifies the curved physical cuts observed in [44] which appear when the matrix-model potentials are perturbed with complex coefficients.²² This consideration is further extended to *off-shell backgrounds (or spectral curves)* in terms of the Riemann-Hilbert approach in Section 5.

- In the $\hat{p} > 1$ cases, the exponents $\varphi^{(j)}(t; \zeta)$ have non-trivial cuts in the ζ plane, say $\varphi^{(j)}(t; \zeta) \sim \zeta^{(\hat{q}+1)/\hat{p}}$. This \hat{p} -th root cut should be smeared by a proper supplement

²²This consideration suggests that the position of physical cuts are not freely assigned and closely related to non-perturbative consistency with Stokes phenomena and therefore with D-instanton chemical potentials.

of exponents [52]. This is also reviewed in Appendix A. Since we concentrate on the $\hat{p} = 1$ cases in this paper, this point in the general k -cut cases remains to be studied for future investigations.

- As it will be clear in Section 4.2.2, the set of indices (a_1, a_2, \dots, a_k) in the multi-cut boundary condition (Definition 10) is generally given as

$$a_n = j_{2r(n-1),k} = n + (n-1)(r-\gamma)m_1, \quad (4.20)$$

with Theorem 3. In particular, the \mathbb{Z}_k symmetric cases ($\gamma = r$) is given as $a_n = n$.

Next we apply this boundary condition to concrete systems. Before devoting ourselves into general cases, however, we first consider the multi-cut boundary condition in the two-cut case, as a warm-up exercise for the general systems.

4.2.1 The two-cut boundary condition

Here we show how to solve the multi-cut boundary conditions in the two-cut $(1, 2)$ case. The orthonormal polynomial $\Psi_{\text{orth}}(t; \zeta)$ in a Stokes sector D_n is generally given as a superposition of independent solutions, $\tilde{\Psi}_n^{(j)}(t; \zeta)$:

$$\Psi_{\text{orth}}(t; \zeta) = \tilde{\Psi}_n(t; \zeta) X^{(n)} = x_1^{(n)} \tilde{\Psi}_n^{(1)}(t; \zeta) + x_2^{(n)} \tilde{\Psi}_n^{(2)}(t; \zeta), \quad \zeta \rightarrow \infty \in D_n. \quad (4.21)$$

However this assumption results in the 6-cut geometry of resolvent as shown in Fig. 8-a, even though this system is called “two-cut”. Therefore, one has to choose proper Stokes multipliers in order to satisfy the multi-cut boundary condition and therefore to obtain the geometry which only includes two cuts as shown in Fig. 8-b.

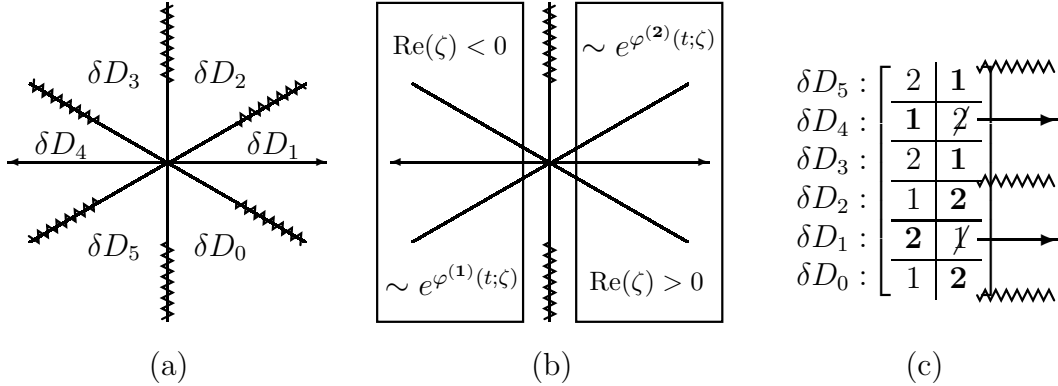


Figure 8: The positions of cuts in the two-cut $(1, 2)$ ODE system. a) A general configuration of cuts for the general Stokes multipliers. There are 6 cuts. b) A configuration of cuts for the $(1, 2)$ critical point in the two-cut matrix models. The boxes indicate the regions $\text{Re}(\zeta) > 0$ and $\text{Re}(\zeta) < 0$, in which the asymptotic expansion is given by $\sim e^{\varphi^{(i)}(\zeta)}$ ($i = 1, 2$). c) The profile of dominance depicted with the position of cuts and the weak coupling infinity $\zeta \rightarrow \pm\infty \in \mathbb{R}$.

The multi-cut boundary condition is then given as follows: Since we wish to erase the cuts of orthonormal polynomial (4.10) along the Stokes lines of

$$\theta = \pm \frac{\pi}{3}, \quad \pm \frac{5\pi}{3}, \quad (4.22)$$

we impose the following boundary condition on the vectors $X^{(n)}$:

$$\begin{aligned} X^{(0)} &= \begin{pmatrix} 0 \\ x_2^{(0)} \end{pmatrix}, & X^{(1)} &= \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix}, & X^{(2)} &= \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}, \\ X^{(3)} &= \begin{pmatrix} x_1^{(3)} \\ 0 \end{pmatrix}, & X^{(4)} &= \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix}, & X^{(5)} &= \begin{pmatrix} x_1^{(5)} \\ x_2^{(5)} \end{pmatrix}, \end{aligned} \quad (4.23)$$

where all the $x_i^{(n)}$ appearing here are non-zero. This can be also expressed in the dominance profile as in Fig. 8-c. That is, if the Stokes sector D_n includes the following profile,

$$[m_1 \mid \cdots \mid m_{I-1} \mid \mathbf{m}_I \mid \not{m}_{I+1} \mid \cdots \mid \not{m}_{k-1} \mid \not{m}_k] \in D_n, \quad (4.24)$$

then the boundary condition can be read as

$$\Psi_{\text{orth}}(t; \zeta) = \sum_{j=1}^I x_{m_j}^{(n)} \tilde{\Psi}_n^{(m_j)}(t; \zeta), \quad x_{m_I}^{(n)} \neq 0. \quad (4.25)$$

Since these vectors are related with the Stokes matrix (2.54) (with the \mathbb{Z}_2 symmetry condition (2.59)) as

$$X^{(n)} = S_n X^{(n+1)}, \quad X^{(n+6)} = X^{(n)}, \quad (4.26)$$

one obtains the following conditions on the vectors $X^{(n)}$ and the Stokes multipliers:

$$\begin{aligned} \begin{pmatrix} 0 \\ x_2^{(0)} \end{pmatrix} &= \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix}, & \begin{pmatrix} 0 \\ x_2^{(1)} \end{pmatrix} &= \begin{pmatrix} x_1^{(2)} + \beta x_2^{(2)} \\ x_2^{(2)} \end{pmatrix}, & \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix} &= \begin{pmatrix} x_1^{(3)} \\ \gamma x_1^{(3)} \end{pmatrix}, \\ \begin{pmatrix} x_1^{(3)} \\ 0 \end{pmatrix} &= \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix}, & \begin{pmatrix} x_1^{(4)} \\ 0 \end{pmatrix} &= \begin{pmatrix} x_1^{(5)} \\ \beta x_1^{(5)} + x_2^{(5)} \end{pmatrix}, & \begin{pmatrix} x_1^{(5)} \\ x_2^{(5)} \end{pmatrix} &= \begin{pmatrix} \gamma x_2^{(0)} \\ x_2^{(0)} \end{pmatrix}, \end{aligned} \quad (4.27)$$

which results in

$$\begin{aligned} \beta^2 &= 1, & \gamma^2 &= 1, & 1 + \beta\gamma &= 0, \\ x_1^{(2)} &= x_1^{(3)} = x_1^{(4)} = x_1^{(5)} = \gamma x_2^{(0)} \neq 0, & x_2^{(5)} &= x_2^{(0)} = x_2^{(1)} = x_2^{(2)} = \gamma x_1^{(3)} \neq 0. \end{aligned} \quad (4.28)$$

Therefore, the solutions which are consistent with the Hermiticity condition (2.63) and with the monodromy free condition (2.66) are given as

$$\alpha \in i\mathbb{R}, \quad \beta = -\gamma = \pm 1. \quad (4.29)$$

Consequently, the solution to the multi-cut boundary condition in the two-cut case has a real continuum parameter. However, as we will discuss in Section 5 with the Riemann-Hilbert approach, this parameter α causes “exponentially growing non-perturbative corrections” to the perturbative backgrounds (e.g. one-cut/two-cut spectral curves), except when $\alpha = 0$. Therefore, the multi-cut boundary condition (with “the small instanton condition”) completely fix the D-instanton chemical potentials as we advertised at the end of Section 2.

4.2.2 The multi-cut boundary-condition recursions ($r = 2$)

From here, we solve the multi-cut boundary condition for an arbitrary number of cuts, k . In order to solve the constraints, we use the symmetric Stokes sectors (See Definition 8),

$$\Psi_{\text{orth}}(t; \zeta) \underset{\text{asym}}{\simeq} \tilde{\Psi}_{2rl}(t; \zeta) X^{(2rl)}, \quad \zeta \rightarrow \infty \in D_{2rl}, \quad (l = 0, 1, 2, \dots, k-1), \quad (4.30)$$

and its Stokes matrices, $S_{2rl}^{(\text{sym})} = \Gamma^{-l} S_0^{(\text{sym})} \Gamma^l$. For sake of simplicity, however, we here focus on the $r = 2$ cases, and therefore $k = 5, 7, 9, \dots$.²³ Some of the results can be easily generalized to the general r cases.

We first read the boundary condition in terms of the dominance profile:

Proposition 1 (The multi-cut boundary condition on $X^{(n)}$) *The multi-cut boundary condition in the general k -cut cases with $r = 2$ is given as*

The general k -cut cases

$k = 4k_0 + 1 \quad D_{2rn} :$

$$\left[\begin{array}{c|c|c} \dots & n + \frac{k+5}{2} + \lfloor \frac{k-3}{4} \rfloor & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor \\ \dots & n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor \\ \dots & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor & \not n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor \\ \dots & \mathbf{n} + \lfloor \frac{k-1}{4} \rfloor & \not n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor \\ \vdots & \vdots & \vdots \\ \dots & n + \frac{k+5}{2} & \mathbf{n} + 2 \\ \dots & n + \frac{k+3}{2} & \mathbf{n} + 2 \\ \dots & \mathbf{n} + 2 & \not n + \frac{k+3}{2} \\ \dots & \mathbf{n} + 1 & \not n + \frac{k+3}{2} \\ \dots & n + \frac{k+3}{2} & \mathbf{n} + 1 \\ \dots & n + \frac{k+1}{2} & \mathbf{n} + 1 \end{array} \right],$$

$k = 4k_0 + 3 \quad D_{2rn} :$

$$\left[\begin{array}{c|c|c} \dots & \mathbf{n} + \lfloor \frac{k+7}{4} \rfloor & \not n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor \\ \dots & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor & \not n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor \\ \dots & n + \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor \\ \dots & n + \frac{k+1}{2} + \lfloor \frac{k-3}{4} \rfloor & \mathbf{n} + \lfloor \frac{k+3}{4} \rfloor \\ \vdots & \vdots & \vdots \\ \dots & n + \frac{k+5}{2} & \mathbf{n} + 2 \\ \dots & n + \frac{k+3}{2} & \mathbf{n} + 2 \\ \dots & \mathbf{n} + 2 & \not n + \frac{k+3}{2} \\ \dots & \mathbf{n} + 1 & \not n + \frac{k+3}{2} \\ \dots & n + \frac{k+3}{2} & \mathbf{n} + 1 \\ \dots & n + \frac{k+1}{2} & \mathbf{n} + 1 \end{array} \right].$$

(4.31)

Equivalently, the components of $X^{(4n)}$ ($r = 2$ and $k \geq 5$) is given as

$$\begin{aligned} x_{n+i}^{(4n)} &\neq 0 & (i = 1, 2, \dots, \lfloor \frac{k+3}{4} \rfloor), \\ x_{n+\frac{k+1}{2}+i}^{(4n)} &= 0 & (i = 1, 2, \dots, \lfloor \frac{k+1}{4} \rfloor), \end{aligned} \quad (4.32)$$

for $n = 0, 1, 2, \dots, k-1$. The constraints on the Stokes matrices are then imposed by Eq. (4.11).

It is then convenient to introduce a new vector $Y^{(4n)} = (y_{n,j})_{j=1}^k \equiv \Gamma^n X^{(4n)}$, since the

²³Here $k = 3$ is special because $k < 2r = 4$. This case is calculated separately in Appendix E.

above boundary condition becomes simpler:

$$X^{(4n)} = \begin{pmatrix} \vdots \\ x_{n+1}^{(4n)} \neq 0 \\ \vdots \\ x_{n+\lfloor \frac{k+3}{4} \rfloor}^{(4n)} \neq 0 \\ \vdots \\ x_{n+\frac{k+3}{2}}^{(4n)} = 0 \\ \vdots \\ x_{n+\frac{k+3}{2}+\lfloor \frac{k-3}{4} \rfloor}^{(4n)} = 0 \\ \vdots \end{pmatrix}, \quad Y^{(4n)} = \begin{pmatrix} y_{n,1} \neq 0 \\ \vdots \\ y_{n,\lfloor \frac{k+3}{4} \rfloor} \neq 0 \\ \vdots \\ y_{n,\frac{k+3}{2}} = 0 \\ \vdots \\ y_{n,\frac{k+3}{2}+\lfloor \frac{k-3}{4} \rfloor} = 0 \\ \vdots \end{pmatrix}. \quad (4.33)$$

Note that the periodicity of index n follows:

$$X^{(4n)} = X^{(4(n+k))}, \quad Y^{(4n)} = Y^{(4(n+k))}, \quad y_{n+k,j} = y_{n,j}. \quad (4.34)$$

In terms of the vector $Y^{(4n)}$, the constraints on the Stokes multipliers Eq. (4.11) are expressed as

$$X^{(4n)} = S_{4n}^{(\text{sym})} X^{(4(n+1))} \Leftrightarrow Y^{(4n)} = (S_0^{(\text{sym})} \Gamma^{-1}) Y^{(4(n+1))}. \quad (4.35)$$

Therefore, in terms of components, we obtain the following recursive relations for $y_{n,i}$:

$$y_{n,i} = y_{n+1,i-1} + \sum_{j=1}^k s_{0,i,j}^{(\text{sym})} y_{n+1,j-1}, \quad y_{n+k,j} = y_{n,j}. \quad (4.36)$$

This is the central equations for the multi-cut boundary condition. After some tedious calculations, the multi-cut boundary condition turns out to be the following simple form:

Theorem 5 (The multi-cut BC recursions) *The recursion relation Eq. (4.36) with the multi-cut boundary condition Eq. (4.33) in the $(k, r; \gamma) = (2m+1, 2; 2)$ case is equivalent to the following two recursion equations for $\{y_{n,1}\}_{n \in \mathbb{Z}}$:*

$$\begin{aligned} \mathcal{F}_k[y_{n,1}] &= y_{n+m,1} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} s_{1,m+2-j,1+j} \times y_{n+2j-1,1} + \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} s_{3,m+3-j,1+j} \times y_{n+2j-2,1} = 0, \\ \mathcal{G}_k[y_{n,1}] &= -y_{n,1} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} s_{0,k+1-j,1+j} \times y_{n+2j,1} + \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} s_{2,k+2-j,1+j} \times y_{n+2j-1,1} = 0, \end{aligned} \quad (4.37)$$

and linear expressions of the components $\{y_{n,i}\}_{1 \leq i \leq k}^{n \in \mathbb{Z}}$ in terms of $\{y_{n,1}\}_{n \in \mathbb{Z}}$:

$$y_{n,i} = y_{n,i}(\{y_{l,1}\}_{l \in \mathbb{Z}}). \quad (4.38)$$

Note that the coefficients in Eqs. (4.37) are understood as modulo k , say $s_{2,i,j} = s_{2,i+k,j}$.

The explicit expression for Eq. (4.38) is a bit long and therefore shown in Appendix C with some examples. These recursive equations are the physical constraints which should be solved in addition to the basic constraints discussed in Section 3.3. In the general cases, the vectors $Y^{(n)}$ in terms of $\{y_{n,1}\}_{n \in \mathbb{Z}}$ are denoted as

$$Y^{(n)}(\{y_{l,1}\}_{l \in \mathbb{Z}}) \equiv \left(y_{n,i}(\{y_{l,1}\}_{l \in \mathbb{Z}}) \right)_{i=1}^k. \quad (4.39)$$

An important point is that all the Stokes multipliers $s_{l,i,j}$ in this expression are *fine Stokes multipliers*. Some detail derivation of this theorem can be found in [102].

Finally we also make a comment on the boundary condition for general $r (= 2, 3, \dots)$. In terms of the dominance profile, they are expressed as

$$D_{2r(n-1)} \supset \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & \vdots & \vdots \\ \hline & & & & & A_3^{(n+1)} & \mathbf{n} + \mathbf{1} \\ \hline & & & & & A_{2r-1}^{(n)} & \mathbf{n} + \mathbf{1} \\ \hline & & & & & \mathbf{n} + \mathbf{1} & A_{2r-1}^{(n)} \\ \hline & & & & \mathbf{n} + \mathbf{1} & A_{2r-3}^{(n)} & A_{2r-1}^{(n)} \\ \hline & & & \mathbf{n} + \mathbf{1} & A_{2r-5}^{(n)} & A_{2r-1}^{(n)} & A_{2r-3}^{(n)} \\ \hline & & \ddots & \ddots & \ddots & & \vdots \\ \hline & \mathbf{n} + \mathbf{1} & A_5^{(n)} & A_{2r-1}^{(n)} & & & \vdots \\ \hline \mathbf{n} + \mathbf{1} & A_3^{(n)} & A_{2r-1}^{(n)} & & & & \vdots \\ \hline \mathbf{n} & A_{2r-1}^{(n)} & A_3^{(n)} & & & & \vdots \\ \hline & \mathbf{n} & A_{2r-3}^{(n)} & A_3^{(n)} & & & \vdots \\ \hline & & \ddots & \ddots & \ddots & & \vdots \\ \hline & & & \mathbf{n} & A_7^{(n)} & A_3^{(n)} & A_5^{(n)} \\ \hline & & & & \mathbf{n} & A_5^{(n)} & A_3^{(n)} \\ \hline & & & & & \mathbf{n} & A_3^{(n)} \\ \hline & & & & & A_3^{(n)} & \mathbf{n} \\ \hline & & & & & A_{2r-1}^{(n-1)} & \mathbf{n} \\ \hline \end{array}, \quad (4.40)$$

Here we define

$$A_i^{(n)} \equiv j_{i+2r(n-1),k} = n + \lfloor i/2 \rfloor m_1 \quad (4.41)$$

with Theorem 3. Note that $A_{i+2r}^{(n)} = A_i^{(n+1)}$ and $A_0^{(n)} = A_1^{(n)} = n$. Therefore if the Stokes sector $D_{2r(n-1)}$ includes the indices $A_i^{(n)}$ of $i = 3, 5, \dots, 2r-1, 2r+3, 2r+5, \dots$, in the profiles, then we impose

$$x_{A_i^{(n)}}^{(2r(n-1))} = 0 \quad (i = 3, 5, \dots) \quad \text{and} \quad x_{n+i}^{(2r(n-1))} \neq 0 \quad (i = 0, 1, \dots). \quad (4.42)$$

The ending points of these series (about i) depend on how many segments $D_{2r(n-1)}$ includes. This general classification could be tedious and we shall leave it for future study.

4.2.3 The complementary boundary conditions

It is suggestive to show which Stokes multipliers appear in the recursive equations Eqs. (4.37). Here we show them by bold type in the profile of $\mathcal{J}_{k,2}^{(\text{sym})}$ (i.e. Theorem 4):

$$\begin{array}{c}
 \dots \mid k-1 \mid \quad (4 \mid \frac{k-1}{2}) \mid \quad (\frac{k+7}{2} \mid k) \mid \quad (3 \mid \frac{k+1}{2}) \mid \quad (\frac{k+5}{2} \mid 1) \mid \quad (2 \mid \frac{k+3}{2}) \\
 \dots \mid (4 \mid \mathbf{k}-1) \mid \quad (\frac{k+7}{2} \mid \frac{k-1}{2}) \mid \quad (3 \mid \mathbf{k}) \mid \quad (\frac{k+5}{2} \mid \frac{k+1}{2}) \mid \quad (2 \mid 1) \mid \quad (\frac{k+3}{2} \mid \frac{k+3}{2}) \\
 \dots \mid \frac{k-3}{2} \mid \quad (\frac{k+7}{2} \mid k-1) \mid \quad (3 \mid \frac{k-1}{2}) \mid \quad (\frac{k+5}{2} \mid k) \mid \quad (2 \mid \frac{k+1}{2}) \mid \quad (\frac{k+3}{2} \mid 1) \\
 \dots \mid (\frac{k+7}{2} \mid \frac{k-3}{2}) \mid \quad (3 \mid \mathbf{k}-1) \mid \quad (\frac{k+5}{2} \mid \frac{k-1}{2}) \mid \quad (2 \mid \mathbf{k}) \mid \quad (\frac{k+3}{2} \mid \frac{k+1}{2}) \mid \quad 1
 \end{array} \begin{array}{l} : 3 \\ : 2 \\ : 1 \\ : 0 \end{array} . \quad (4.43)$$

Note that exactly a half of multipliers $s_{l,i,j} \leftrightarrow (j|i)_l \in \mathcal{J}_{k,2}^{(\text{sym})}$ are picked up by the recursion. One may have the following question: *Are there similar equations which pick up exactly another half of the multipliers?* This can be positively answered. In general, we can expect that there are $(r-1)$ similar equations, each of which picks up a different set of Stokes multipliers.²⁴ These equations come from *complementary boundary conditions* which are given by the multi-cut boundary condition (Definition 10) with different initial angles χ_0 :

$$\chi_0 = \frac{\pi}{k} + \frac{2\pi a}{rk} \quad (a = 1, 2, \dots, r-1). \quad (4.44)$$

The case of $a = 0$ is the original multi-cut boundary condition Eq. (4.18). In fact, in the $r = 2$ case, the vectors $Y^{(4n)}$ in the recursion equation Eq. (4.35) are replaced by the following $\tilde{Y}^{(4n)}$

$$\tilde{Y}^{(4n)} = \begin{pmatrix} \tilde{y}_{n,1} = 0 \\ \vdots \\ \tilde{y}_{n, \lfloor \frac{k+3}{4} \rfloor} = 0 \\ \vdots \\ \tilde{y}_{n, \frac{k+3}{2}} \neq 0 \\ \vdots \\ \tilde{y}_{n, \frac{k+3}{2} + \lfloor \frac{k-3}{4} \rfloor} \neq 0 \\ \vdots \end{pmatrix}, \quad (4.45)$$

satisfying the complementary boundary condition of Eq. (4.44) ($a = 1$), and consequently we found the following different recursion equations with $a = 1$:

Theorem 6 (The complementary BC recursion equations) *The recursion relation Eq. (4.36) with the complementary boundary condition ($a = 1$ of Eq. (4.44)) in the $(k, r; \gamma) = (2m+1, 2; 2)$ case is equivalent to the following two recursion equations for*

²⁴This anticipation is further explicitly shown in the fractional-superstring critical points ($\gamma = r-2$) with arbitrary Poincaré index r [103].

$\{\tilde{y}_{n,m+2}\}_{n \in \mathbb{Z}}$:

$$\begin{aligned} \tilde{\mathcal{F}}_k[\tilde{y}_{n,m+2}] &= \tilde{y}_{m+n,m+2} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} s_{3,k+2-j,m+2+j} \times \tilde{y}_{n+2j-1,m+2} + \\ &\quad + \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} s_{1,k+2-j,m+1+j} \times \tilde{y}_{n+2j-2,m+2} = 0, \end{aligned} \quad (4.46)$$

$$\begin{aligned} \tilde{\mathcal{G}}_k[\tilde{y}_{n,m+2}] &= -\tilde{y}_{n,m+2} + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor} s_{2,m+2-j,m+2+j} \times \tilde{y}_{n+2j,m+2} + \\ &\quad + \sum_{j=1}^{\lfloor \frac{m+1}{2} \rfloor} s_{0,m+2-j,m+1+j} \times \tilde{y}_{n+2j-1,m+2} = 0, \end{aligned} \quad (4.47)$$

and linear expressions of the components $\{\tilde{y}_{n,i}\}_{1 \leq i \leq k}^{n \in \mathbb{Z}}$ in terms of $\{\tilde{y}_{n,m+2}\}_{n \in \mathbb{Z}}$:

$$\tilde{y}_{n,i} = \tilde{y}_{n,i}(\{\tilde{y}_{l,m+2}\}_{l \in \mathbb{Z}}). \quad (4.48)$$

Note that the coefficients in Eqs. (4.47) are understood as modulo k , say $s_{2,i,j} = s_{2,i+k,j}$.

The explicit expression for Eq. (4.48) is a bit long and therefore shown in Appendix C. The vectors $\tilde{Y}^{(n)}$ in terms of $\{\tilde{y}_{n,m+2}\}_{n \in \mathbb{Z}}$ are also denoted by

$$\tilde{Y}^{(n)}(\{\tilde{y}_{l,m+2}\}_{l \in \mathbb{Z}}) = \left(\tilde{y}_{n,i}(\{\tilde{y}_{l,m+2}\}_{l \in \mathbb{Z}}) \right)_{i=1}^k, \quad (4.49)$$

In these recursion equations, exactly the complementary set of the multipliers $s_{l,i,j} \leftrightarrow (j|i)_l \in \mathcal{J}_{k,2}^{(\text{sym})}$ are picked up.

4.2.4 Useful reparametrization of the Stokes multipliers

It is again suggestive to express these recursion equations as follows:²⁵

$$\mathcal{F}_k[y_{n-m,1}] = y_{n,1} + \sum_{i=1}^m \theta_i y_{n-i,1} = 0,$$

²⁵We point out the following interesting facts about these expressions. The algebraic equations defined by the recursion equations Eq. (4.52),

$$\begin{aligned} \mathcal{F}_k(y) &\equiv y^{-n} \mathcal{F}_k[\{y_{j,1} \rightarrow y^j\}_{j \in \mathbb{Z}}] = y^m + \sum_{n=1}^m \theta_n y^{m-n} = 0, \\ \mathcal{G}_k(y) &\equiv y^{-n} \mathcal{G}_k[\{y_{j,1} \rightarrow y^j\}_{j \in \mathbb{Z}}] = -\left(1 + \sum_{n=1}^m \theta_n^* y^n\right) = 0 \end{aligned} \quad (4.50)$$

satisfy the following hermiticity relation:

$$[\mathcal{F}_k(y)]^* = -y^{-m} \mathcal{G}_k(y), \quad \text{if} \quad y^k = 1. \quad (4.51)$$

The same thing also happen for Eq. (4.53). This therefore suggests that the solutions to the recursions $\{y_{n,1}\}_{n \in \mathbb{Z}}$ are given by k -th roots of unity, $y^k = 1$.

$$\mathcal{G}_k[y_{n,1}] = -\left(y_{n,1} + \sum_{i=1}^m \theta_i^* y_{n+i,1}\right) = 0; \quad (4.52)$$

$$\begin{aligned} \tilde{\mathcal{F}}_k[\tilde{y}_{n-m,m+2}] &= \tilde{y}_{n,m+2} + \sum_{i=1}^m \tilde{\theta}_i \tilde{y}_{n-i,m+2} = 0, \\ \tilde{\mathcal{G}}_k[\tilde{y}_{n,m+2}] &= -\left(\tilde{y}_{n,m+2} + \sum_{i=1}^m \tilde{\theta}_i^* \tilde{y}_{n+i,m+2}\right) = 0. \end{aligned} \quad (4.53)$$

The complex conjugation θ_n^* (and $\tilde{\theta}_n^*$) comes from the hermiticity condition of Stokes multipliers Eq. (3.47). It is also interesting to see the index n of the parameters θ_n (and $\tilde{\theta}_n$) in terms of the dominance profile:

$$\left. \begin{array}{l} \cdots \quad m-5 \quad)(\quad \mathbf{m}-4 \quad)(\quad m-3 \quad)(\quad \mathbf{m}-2 \quad)(\quad m-1 \quad)(\quad \mathbf{m} \quad) \\ \cdots \quad)(\quad \mathbf{5} \quad)(\quad 4 \quad)(\quad \mathbf{3} \quad)(\quad 2 \quad)(\quad \mathbf{1} \quad) \\ \cdots \quad \mathbf{m}-5 \quad)(\quad m-4 \quad)(\quad \mathbf{m}-3 \quad)(\quad m-2 \quad)(\quad \mathbf{m}-1 \quad)(\quad m \quad) \\ \cdots \quad)(\quad 5 \quad)(\quad \mathbf{4} \quad)(\quad 3 \quad)(\quad \mathbf{2} \quad)(\quad 1 \quad) \end{array} \right] : \begin{array}{l} 3 \\ 2 \\ 1 \\ 0 \end{array} \quad (4.54)$$

Here bold type is again the coefficients of the multi-cut BC recursions Eqs. (4.37). An important thing here is that these complementary boundary conditions are used to obtain explicit solutions of the Stokes multipliers (although they are not related to the physical boundary conditions).

Finally, in order to write the explicit relation between the Stokes multipliers and the parameters θ_n , we introduce integers $L_{l,i,j}$ ($l, i, j \in \mathbb{Z}$) which are defined as

$$0 \leq L_{l,i,j} < k, \quad L_{l,i,j} \equiv (-1)^{l-1}(i-j) \pmod{k}. \quad (4.55)$$

In particular, we pick up the following set of indices $(l; i, j)$:

$$\underline{k = 4k_0 + 1} : \quad L_{l,i,j} + \left\lfloor \frac{l-1}{2} \right\rfloor \in 2\mathbb{Z} + 1; \quad \underline{k = 4k_0 + 3} : \quad L_{l,i,j} + \left\lfloor \frac{l}{2} \right\rfloor \in 2\mathbb{Z}, \quad (4.56)$$

and the relation is given as follows:

Proposition 2 (The θ_n parametrization) *The fine Stokes multipliers $s_{l,i,j}$ are parametrized by $k-1$ complex parameters $\{\theta_n, \tilde{\theta}_n\}_{n=1}^{\lfloor \frac{k}{2} \rfloor}$ as*

$$(j|i) \in \mathcal{J}_{k,2}^{(\text{sym})} \text{ satisfying Eq. (4.56)} : \quad s_{l,i,j} = \begin{cases} \theta_{L_{l,i,j}} & (l = 1, 3) \\ -\theta_{L_{l,i,j}}^* & (l = 0, 2) \end{cases}, \quad (4.57)$$

and

$$(j|i) \in \mathcal{J}_{k,2}^{(\text{sym})} \text{ not satisfying Eq. (4.56)} : \quad s_{l,i,j} = \begin{cases} \tilde{\theta}_{L_{l,i,j}} & (l = 1, 3) \\ -\tilde{\theta}_{L_{l,i,j}}^* & (l = 0, 2) \end{cases}. \quad (4.58)$$

Therefore, this is a one to one correspondence up to the hermiticity condition Eq. (4.62).

4.3 Solutions in the general k -cut cases

Before solving the boundary conditions, here we summarize the equations to be solved: After imposing the \mathbb{Z}_k symmetry condition (3.39),

$$\underline{\mathbb{Z}_k \text{ symmetry:}} \quad S_{2rl}^{(\text{sym})} = \Gamma^{-l} S_0^{(\text{sym})} \Gamma^l, \quad (l = 1, 2, \dots, k-1) \quad (4.59)$$

the system becomes

$$\underline{\text{Multi-cut BC recursion:}} \quad Y^{(4n)} = (S_0^{(\text{sym})} \Gamma^{-1}) Y^{(4(n+1))} \quad (4.60)$$

$$\underline{\text{Monodromy free condition:}} \quad (S_0^{(\text{sym})} \Gamma^{-1})^k = I_k, \quad (4.61)$$

$$\underline{\text{Hermiticity condition:}} \quad S_n^* = \Delta \Gamma S_{(2r-1)k-n}^{-1} \Gamma^{-1} \Delta, \quad (4.62)$$

for $n = 0, 1, 2, \dots, k-1$. In general, the patterns of solutions become complicated if we increase the number of cuts. However, we here show two kinds of special solutions which can be generalized to the cases with an arbitrary number of cuts.

Before showing explicit solutions, we mention a key point of solving the above equations. The main difficulty is from the monodromy free condition (4.61). We here note that following fact:²⁶

Lemma 1 *If the matrix $S_0^{(\text{sym})} \Gamma^{-1}$ is diagonalizable and its eigenvalues λ_j are k -th roots of unity $\lambda_j^k = 1$, then the monodromy free condition (4.61) is satisfied. The opposite is also true.*

Our strategy of finding solutions is now to show that $S_0^{(\text{sym})} \Gamma^{-1}$ is diagonalizable. Below we list two types of explicit solutions:

4.3.1 Discrete solutions and configurations of avalanches

Theorem 7 (Discrete Solution) *The following Stokes multipliers $s_{i,j}$ (written with $\theta_n, \tilde{\theta}_n$ of Proposition 2) are solutions to the multi-cut boundary condition in the \mathbb{Z}_k symmetric $(\hat{p}, \hat{q}) = (1, 1)$ k -cut critical points ($k = 2m + 1, \gamma = r = 2$):*

$$\theta_n = \sigma_n(\{-\omega^{n_j}\}_{j=1}^m), \quad \tilde{\theta}_n = \sigma_n(\{-\omega^{\tilde{n}_j}\}_{j=1}^m), \quad (n = 1, 2, \dots, m) \quad (4.63)$$

with the symmetric polynomials σ_n among $\{x_i\}_{i=1}^N$ of degree n .²⁷

$$\sigma_n(\{x_i\}_{i=1}^N) \equiv \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} x_{i_1} x_{i_2} \dots x_{i_n}, \quad (4.64)$$

if and only if the integers $(n_1, n_2, \dots, n_m; \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_m)$ of Eq. (4.63) satisfy

$$\begin{cases} n_i \not\equiv n_j, & \tilde{n}_i \not\equiv \tilde{n}_j \pmod{k} & (i \neq j), \\ -\sum_{j=1}^m (n_j + \tilde{n}_j) \not\equiv n_i, & \tilde{n}_i \pmod{k} & (i = 1, 2, \dots, m), \end{cases} \quad (4.65)$$

²⁶The opposite is non-trivial but can be shown by using Jordan normal form.

²⁷Here σ_n stands for the symmetric polynomials. Do not be confused with the Pauli matrices.

Comments on the conditions Eqs. (4.65) are following:

- In this solution, one can find the $(k-1)$ explicit eigenvectors of the matrix $S_0^{(\text{sym})}\Gamma^{-1}$:

$$\omega^{-n_i}\mathcal{Y}_i = (S_0^{(\text{sym})}\Gamma^{-1})\mathcal{Y}_i, \quad \omega^{-\tilde{n}_j}\tilde{\mathcal{Y}}_j = (S_0^{(\text{sym})}\Gamma^{-1})\tilde{\mathcal{Y}}_j, \quad (i, j = 1, 2, \dots, m), \quad (4.66)$$

with the vectors of the BC recursion equations, Eq. (4.39) and Eq. (4.49):

$$\mathcal{Y}_i \equiv Y^{(0)}[\{y_{n,1} \rightarrow \omega^{n \times n_i}\}_{n \in \mathbb{Z}}], \quad \tilde{\mathcal{Y}}_j \equiv \tilde{Y}^{(0)}[\{\tilde{y}_{n,m+2} \rightarrow \omega^{n \times \tilde{n}_j}\}_{n \in \mathbb{Z}}]. \quad (4.67)$$

They are distinct only when $n_i \not\equiv n_j$ ($i \neq j$) and $\tilde{n}_i \not\equiv \tilde{n}_j$ ($i \neq j$) with modulo k .

- Noting that $\det S_0^{(\text{sym})}\Gamma^{-1} = 1$, one concludes that the eigenvalue of the remaining eigenvector is given by ω^{-n_0} with $n_0 \equiv -\sum_{j=1}^m(n_j + \tilde{n}_j)$. This eigenvector becomes distinct only when $n_i \not\equiv n_0 \not\equiv \tilde{n}_j$ ($i, j = 1, 2, \dots, m$).²⁸

With these considerations, one can prove the above theorem. Below we show a graphical expression of the conditions Eq. (4.65) in terms of Young diagram.

1. The following transformation is an automorphism among the solutions to the conditions (4.65):

$$(n_1, n_2, \dots, n_m; \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_m) \rightarrow (n_1 + 1, n_2 + 1, \dots, n_m + 1; \tilde{n}_1 + 1, \tilde{n}_2 + 1, \dots, \tilde{n}_m + 1), \quad (4.69)$$

which also maps n_0 ($\equiv -\sum_{j=1}^m(n_j + \tilde{n}_j)$) as $n_0 \rightarrow n_0 + 1$.

2. By choosing the following representative of the solutions as $n_0 \equiv 0$, and by properly choosing the ordering of the indices, one can rewrite the conditions (4.65) as

$$\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} (n_j + \tilde{n}_j) \equiv 0, \quad \begin{cases} 1 \leq n_1 < n_2 < \dots < n_{\lfloor \frac{k}{2} \rfloor} \leq k-1, \\ 1 \leq \tilde{n}_1 < \tilde{n}_2 < \dots < \tilde{n}_{\lfloor \frac{k}{2} \rfloor} \leq k-1 \end{cases}. \quad (4.70)$$

3. Therefore, these indices can be expressed in terms of Young diagram. Here is an example (a solution in the 11-cut case).

$$(n_1, n_2, n_3, n_4, n_5) = (1, 2, 4, 6, 9) \quad \Leftrightarrow \quad \begin{array}{c} 9 \\ 6 \\ 4 \\ 2 \\ 1 \end{array} \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline \end{array}. \quad (4.71)$$

That is, the i -th row from the bottom has n_i sky boxes in the diagram.

²⁸As a side remark, here we show the eigenvector \mathcal{Y}_0 of the eigenvalue $\eta^{-1} \equiv \omega^{-n_0}$ when $n_j = \tilde{n}_j$ ($j = 1, 2, \dots, m$):

$$\begin{aligned} \mathcal{Y}_0 &\equiv Y^{(0)}(\{y_{j,1} \rightarrow \eta^j\}_{j \in \mathbb{Z}}) + \tilde{Y}^{(0)}(\{\tilde{y}_{j,m+2} \rightarrow \eta^j(-1)^m \eta^{1/2}\}_{j \in \mathbb{Z}}) \\ &= \left(1, \eta, \dots, \eta^{\lfloor \frac{m}{2} \rfloor}, \underbrace{0, \dots, 0}_{\text{II}}, (-1)^m \eta^{1/2}, \dots, (-1)^m \eta^{1/2 + \lfloor \frac{m-1}{2} \rfloor}, \underbrace{0, \dots, 0}_{\text{IV}}\right)^t. \end{aligned} \quad (4.68)$$

Note that all the components of \mathcal{Y}_0 in the region II and region IV vanish (See Eqs. (C.1) for definition of the regions).

We also draw $m \times k$ total boxes for later convenience. In particular, the upper-left Young diagram (written with \boxtimes) is referred to as *sky* and the lower-right Young diagram (written with \square) is as *snow*. The pair $(n_j; \tilde{n}_j)_j$ is denoted as

$$(n_1, n_2, n_3, n_4, n_5; \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4, \tilde{n}_5) = (1, 2, 4, 6, 9; 3, 5, 7, 8, 10)$$
(4.72)

Therefore, the graphical meaning of the conditions Eqs. (4.65) is following:

- The number of the boxes \square (amount of snow) is always a multiple of k , and the following configurations are allowed solutions in the 7-cut case:

(4.73)

- Neither n_i and \tilde{n}_i ($i = 1, 2, \dots, m$) can be 0 or k , therefore the following configurations are not allowed:

forbidden:

(4.74)

- The solutions cannot have vertical cliffs, therefore the following configurations are not allowed:

forbidden:

(4.75)

One of the ways to exhaust the solutions is first to take the most steepest configurations, and then to consider possible ways for snow to slide on the surface with satisfying the condition (4.70), for example:

(4.76)

We refer to these configurations of Young diagram as *avalanches*. Therefore, we conclude:

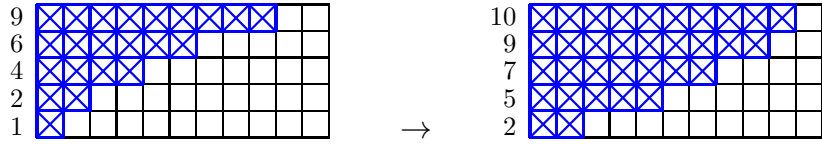
Proposition 3 (Avalanches) *The discrete solutions to the non-perturbative completion are labeled by configurations of avalanches in terms of Young diagram.*

Note that one can also move some snow on the one side to the other side. It is also worth mentioning the following two transformations:

- The first transformation is called *dual*,

$$(n_1, n_2, \dots, n_m; \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_m) \rightarrow (k - n_1, k - n_2, \dots, k - n_m; k - \tilde{n}_1, k - \tilde{n}_2, \dots, k - \tilde{n}_m), \quad (4.77)$$

In the terminology of Young diagram, the dual transformation (4.77) exchanges the sky and snow of the left and right Young diagrams simultaneously. In particular, the following diagram shows the action of the dual transformation on the left Young diagram:



$$\rightarrow \quad (4.78)$$

- The following is called *reflection*, which exchanges the snows of the left and right Young diagrams:

$$(n_1, n_2, \dots, n_m; \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_m) \rightarrow (\tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_m; n_1, n_2, \dots, n_m). \quad (4.79)$$

Note that these two transformations also automorphisms which fix the condition (4.70).

4.3.2 Continuum solutions

Theorem 8 (Continuum Solution) *The following Stokes multipliers $s_{l,i,j}$ (written with $\theta_n, \tilde{\theta}_n$ of Proposition 2) are solutions to the multi-cut boundary condition in the \mathbb{Z}_k symmetric $(\hat{p}, \hat{q}) = (1, 1)$ k -cut critical points ($k = 2m + 1, \gamma = r = 2$):*

$$\theta_n = \sigma_n(\{-\omega^{n_j}\}_{j=1}^m), \quad \tilde{\theta}_n = \mathcal{S}_n(\{\theta_j\}_{j=1}^m) + \tilde{\theta}_{m-n+1}^* \theta_m^*, \quad (n = 1, 2, \dots, m), \quad (4.80)$$

with the polynomial $\mathcal{S}_n(x)$ which are defined by the following recursion relation:

$$\mathcal{S}_n(\{x_j\}_{j \in \mathbb{Z}}) = - \sum_{i=1}^n x_i \mathcal{S}_{n-i}(\{x_j\}_{j \in \mathbb{Z}}), \quad \mathcal{S}_0(\{x_j\}_{j \in \mathbb{Z}}) = 1, \quad (4.81)$$

if and only if the integers $(n_1, n_2, \dots, n_{\lfloor \frac{k}{2} \rfloor})$ satisfy $n_i \not\equiv n_j \pmod k$ ($i \neq j$).

A derivation of this solution is shown in Appendix D. The concrete expression of Eq. (4.80) (and therefore the polynomials $\mathcal{S}_n(x)$) is given as

$$\begin{aligned} \tilde{\theta}_1 &= (-\theta_1) + \tilde{\theta}_m^* \theta_m^*, \\ \tilde{\theta}_2 &= (-\theta_2 + \theta_1^2) + \tilde{\theta}_{m-1}^* \theta_m^*, \\ \tilde{\theta}_3 &= (-\theta_3 + 2\theta_1\theta_2 - \theta_1^3) + \tilde{\theta}_{m-2}^* \theta_m^*, \\ \tilde{\theta}_4 &= (-\theta_4 + \theta_2^2 - 3\theta_1^2\theta_2 + 2\theta_1\theta_3 + \theta_1^4) + \tilde{\theta}_{m-3}^* \theta_m^*, \\ \tilde{\theta}_5 &= (-\theta_5 + \theta_4\theta_1 - 3\theta_1^2\theta_3 + \theta_2\theta_3 + 4\theta_1^3\theta_2 - 3\theta_1\theta_2^2 - \theta_1^5) + \tilde{\theta}_{m-4}^* \theta_m^*. \end{aligned} \quad (4.82)$$

It is worth mentioning the relation to the Schur polynomials $P_n(\{x_j\}_{j \in \mathbb{Z}})$:

$$\mathcal{S}_n(\{x_j\}_{j \in \mathbb{Z}}) = P_n(\{y_j\}_{j \in \mathbb{Z}}), \quad x_n = P_n(\{-y_j\}_{j \in \mathbb{Z}}), \quad (4.83)$$

where the Schur polynomials $P_n(\{x_j\}_{j \in \mathbb{Z}})$ are defined as

$$\sum_{n=0}^{\infty} z^n P_n(\{x_j\}_{j \in \mathbb{Z}}) = \exp \left[\sum_{n=1}^{\infty} z^n x_n \right]. \quad (4.84)$$

Note that these solutions includes $m (= \lfloor \frac{k}{2} \rfloor)$ real parameters. Sometimes, eigenvalues of the matrix $S_0^{(\text{sym})} \Gamma^{-1}$ of the discrete solutions are distinct. In this case, such a discrete solution is a special case of the continuum solution. However generally these solutions do not include the discrete solutions in Section 4.3.1, since the discrete solutions generally include degeneracy of eigenvalues which cannot be resolved by these continuum parameters.

5 Stability of perturbative backgrounds

In this section, we briefly review the Riemann-Hilbert approach and the Deift-Zhou method [84–86], and also discuss its physical interpretations in non-critical string theory. In particular, we argue that this procedure implies *an additional physical requirement* about stability of classical (or perturbative) backgrounds. We will see that this constraint results in the proper Stokes multipliers expected in the two-cut $(1, 2)$ critical point. Classical background here means the spectral curves which appear as semi-classical (large N) solutions of matrix models.

The role of the Riemann-Hilbert approach is to obtain the t dependence of physical amplitudes (for example, asymptotic expansion in t) by using an integration expression which can be derived from the ODE system in ζ . The review article [91] contains useful references of the Riemann-Hilbert approach.

Roughly speaking, in the Riemann-Hilbert approach, we first discard the analytic continuity of the canonical solutions (2.38) and keep the form of asymptotic expansion (2.25) in the complex plane \mathbb{C} . In practice, we introduce some Stokes sectors (here we consider fine Stokes sectors) D_n and canonical solutions on them, $\tilde{\Psi}_n(t; \zeta)$. As it has been reviewed in Section 2, these canonical solutions have the same asymptotic expansion in each Stokes sector (2.38) and the difference of these canonical solutions is expressed by Stokes matrices (2.39). Therefore, inside the intersection of two Stokes sectors, we introduce a semi-infinite straight line from the origin, \mathcal{K}_n ,

$$\mathcal{K}_n = \{\zeta = u e^{i\tilde{\chi}_n}; u \in \mathbb{R}_+\} \quad \text{with} \quad \exists \tilde{\chi}_n \in [0, 2\pi) \quad \text{s.t.} \quad \mathcal{K}_n \subset D_n \cap D_{n+1}, \quad (5.1)$$

and define the following new function $\Psi_{\text{RH}}(t; \zeta)$ which is analytic in $\zeta \in \mathbb{C} \setminus \mathcal{K}$ with $\mathcal{K} \equiv \bigcup_n \mathcal{K}_n$:

$$\Psi_{\text{RH}}(t; \zeta) = \tilde{\Psi}_n(t; \zeta) \quad \zeta \in D(\tilde{\chi}_{n-1}, \tilde{\chi}_n), \quad (n = 1, 2, \dots), \quad (5.2)$$

which has the following uniform asymptotic expansion in $\zeta \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n$:

$$\Psi_{\text{RH}}(t; \zeta) \underset{\text{asym}}{\simeq} \tilde{\Psi}_{\text{asym}}(t; \zeta) = \tilde{Y}(t; \zeta) e^{\tilde{\varphi}(t; \zeta)}, \quad \zeta \rightarrow \infty \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n. \quad (5.3)$$

The lines \mathcal{K} is referred to as discontinuity lines, and examples are shown in Fig. 9. Note that the function $\Psi_{\text{RH}}(t; \zeta)$ has enough information to recover all the canonical solutions simply by analytically continuing the argument ζ .

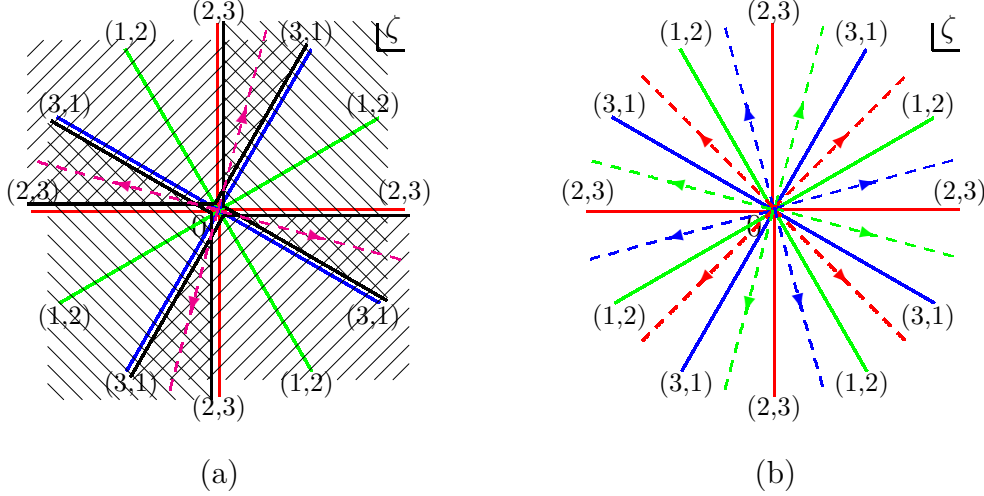


Figure 9: These are examples in the 3-cut (1, 1) critical point. a) The coarse Stokes sectors (shadowed domains) and the discontinuity lines \mathcal{K} (dashed lines). Basically, any lines in the intersections $D_{3n} \cap D_{3(n+1)}$ are allowed. b) The discontinuity lines \mathcal{K} (dashed lines) with respect to the fine Stokes sectors. They are related to the lines in (a) by continuous deformations which do not cross any divergence in the Riemann-Hilbert integral (5.10).

An essence of the Deift-Zhou method for the Riemann-Hilbert problem [86] is introduction of the following $k \times k$ function $g(t; \zeta)$ which we shall call *(off-shell) string background*:

$$g(t; \zeta) = \text{diag}(g^{(1)}(t; \zeta), \dots, g^{(k)}(t; \zeta)),$$

$$\text{with } g^{(i)}(t; \zeta) \equiv \sum_{n=1}^r t_n^{(i)} \zeta^n + t_0^{(i)} \ln \zeta + \sum_{n=0}^{\infty} \frac{1}{n} g_n^{(i)} \zeta^{-n}. \quad (5.4)$$

If one focuses on the aspect of algebraic curves, the function $g(t; \zeta)$ is referred to as *(off-shell) background spectral curve*. We then obtain the following setting of the Riemann-Hilbert problem:

Lemma 2 (Setting of the Riemann-Hilbert problem) *There exists the set of parameters $t_n^{(i)}$ ($i = 1, 2, \dots, k; n = 1, 2, \dots, r$) which satisfies*

$$Z(t; \zeta) \equiv \Psi_{\text{RH}}(t; \zeta) e^{-g(t; \zeta)} \rightarrow I_k, \quad (\zeta \rightarrow \infty \in \mathbb{C} \setminus \bigcup_n \mathcal{K}_n). \quad (5.5)$$

The $k \times k$ matrix function $Z(t; \zeta)$ then satisfies the following discontinuity relation:

$$Z_+(t; \zeta) = Z_-(t; \zeta) G_n(t; \zeta), \quad G_n(t; \zeta) \equiv e^{g(t; \zeta)} S_n e^{-g(t; \zeta)}, \quad \text{along } \zeta \in \mathcal{K}_n, \quad (5.6)$$

where $n = 1, 2, \dots$ and we define $Z_{\pm}(t; \zeta) \equiv \lim_{a \rightarrow 0} Z(t; \zeta \pm a\epsilon)$ with a vector ϵ which directs towards the left hand side of the line \mathcal{K}_n .

In general, the parameters $t_n^{(i)}$ ($i = 1, 2, \dots, k; n = 1, 2, \dots, r$) are the integrable deformations of the k -component KP hierarchy [96]. These are then given by the Lax equations:

$$g_{\text{str}} \frac{\partial}{\partial t_n^{(i)}} \tilde{\Psi}(t; \zeta) = \left[\tilde{\mathcal{P}}_{-n}^{(i)} \zeta^n + \tilde{\mathcal{P}}_{-n+1}^{(i)}(t) \zeta^{n-1} + \dots + \tilde{\mathcal{P}}_0^{(i)} \right] \tilde{\Psi}(t; \zeta) \equiv \tilde{\mathcal{P}}^{(i)}(t; \zeta) \tilde{\Psi}(t; \zeta). \quad (5.7)$$

This information is understood as given information of the system and non-normalizable string moduli space which should not be minimized by the string dynamics [81]. Note that the Stokes matrices are invariants of these integrable deformation:

$$\frac{\partial S_m}{\partial t_n^{(i)}} = 0, \quad (i = 1, \dots, k; n = 1, 2, \dots, r; m = 1, 2, \dots), \quad (5.8)$$

and therefore the multipliers are integration constants (initial conditions) of these deformations. In this sense, they are also understood as non-normalizable string moduli space of the dynamics in the strong-coupling region of string theory.

Since the Stokes multipliers are integration constants of the system, we can uniquely obtain all the information by identifying the deformation parameters $t_n^{(i)}$ ($i = 1, 2, \dots, k; n = 1, 2, \dots, r$) and the Stokes multipliers. The fact is given in the form of the following theorem:

Theorem 9 (The Riemann-Hilbert problem (see [91])) *For a given analytic function $G(t; \zeta)$ on the discontinuity line $\zeta \in \mathcal{K} = \bigcup_n \mathcal{K}_n$,*

$$G(t; \zeta) = G_n(t; \zeta) \equiv e^{g(t; \zeta)} S_n e^{-g(t; \zeta)} \quad \zeta \in \mathcal{K}_n \quad (n = 1, 2, \dots), \quad (5.9)$$

there exists a unique holomorphic function $Z(t; \zeta)$ which satisfies Eqs. (5.5) and (5.6), and is given as

$$\begin{aligned} Z(t; \zeta) &= I_k + \int_{\mathcal{K}} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda)(G(\lambda) - I_k)}{\lambda - \zeta} \\ &= I_k + \sum_{n,i,j} s_{n,i,j} \int_{\mathcal{K}_n} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda) E_{i,j}}{\lambda - \zeta} e^{g^{(i)}(t; \lambda) - g^{(j)}(t; \lambda)}, \end{aligned} \quad (5.10)$$

with $\rho(\zeta) \equiv Z_-(\zeta)$ on $\zeta \in \mathcal{K} = \bigcup_n \mathcal{K}_n$.

By using the Riemann-Hilbert solution (5.10), one can obtain the canonical solutions to the ODE system (defined in (5.3)) as a function of t :

$$\Psi_{\text{RH}}(t; \zeta) = Z(t; \zeta) e^{g(t; \zeta)} \underset{\text{asym}}{\simeq} \tilde{\Psi}_{\text{asym}}(t; \zeta) = \tilde{Y}(t; \zeta) e^{\tilde{\varphi}(t; \zeta)}, \quad \zeta \rightarrow \infty \in \mathbb{C} \setminus \mathcal{K}. \quad (5.11)$$

Note that the “density function $\rho(\zeta)$ ” is given by $Z(t; \zeta)$ itself, and then the function $\rho(\zeta)$ satisfies the following integral equation:

$$\rho(\zeta) = I_k + \int_{\mathcal{K}} \frac{d\lambda}{2\pi i} \frac{\rho(\lambda)(G(\lambda) - I_k)}{\lambda - \zeta + \epsilon}, \quad \zeta \in \mathcal{K}. \quad (5.12)$$

Therefore, one can recursively solve it and the solution is given as the following infinite sum of integrals:

$$Z(t; \zeta) = I_k + \sum_{n=1}^{\infty} \prod_{i=1}^n \left[\int_{\mathcal{K}} \frac{d\lambda_i}{2\pi i} \right] \prod_{j=2}^n \left[\frac{G(\lambda_j) - I_k}{\lambda_j - \lambda_{j-1} + \epsilon} \right] \frac{G(\lambda_1) - I_k}{\lambda_1 - \zeta}, \quad (5.13)$$

with the assumption that

$$\int_{\mathcal{K}} \frac{d\lambda}{2\pi i} (G(\lambda) - I_k), \quad (5.14)$$

is sufficiently small. Note that we use the following multiplication rule of matrices: $\prod_{j=1}^n A_j \equiv A_n A_{n-1} \cdots A_1$. In terms of componets, this is expressed as

$$\begin{aligned} Z(t; \zeta) = & I_k + \sum_{n,i,j} s_{n,i,j} E_{i,j} \int_{\mathcal{K}_n} \frac{d\lambda_1}{2\pi i} \frac{e^{g^{(i)}(t;\lambda_1) - g^{(j)}(t;\lambda_1)}}{\lambda_1 - \zeta} + \\ & + \sum_{n_1, n_2, i, j, l} s_{n_2, i, l} s_{n_1, l, j} E_{i,j} \int_{\mathcal{K}_{n_1}} \frac{d\lambda_1}{2\pi i} \int_{\mathcal{K}_{n_2}} \frac{d\lambda_2}{2\pi i} \frac{e^{g^{(i)}(t;\lambda_2) - g^{(l)}(t;\lambda_2) + g^{(l)}(t;\lambda_1) - g^{(j)}(t;\lambda_1)}}{(\lambda_2 - \lambda_1 - \epsilon)(\lambda_1 - \zeta)} + \cdots \end{aligned} \quad (5.15)$$

This expression is formally convergent if the subsequent integral are small enough. In this case, one can evaluate the leading contribution by truncating higher terms (the so-called Born approximation). It is worth mentioning that this integral is quite similar to the D-instanton operator formalism in the free-fermion formulation [31, 32] by interpreting $g^{(i)}(t; \zeta)$ as the free boson operator $\varphi_0^{(i)}(\zeta)$ in the system.

An important point here is that, in the Riemann-Hilbert approach, the string background $g(t; \zeta)$ is *arbitrary* except for the parameters $t_n^{(i)}$ ($i = 1, 2, \dots, k; n = 1, 2, \dots, r$), and then generally is different from the semi-classical resolvent amplitudes $\tilde{\varphi}(t; \zeta)$ of Eq. (2.32) which is obtained as a solution to the equation of motion (or loop equations) in the large N limit of the matrix models. As one can see in Theorem 9, the role of the string background $g(t; \zeta)$ is a reference background in the Riemann-Hilbert problem. Therefore, from the string-theory viewpoints, the string backgrounds $g(t; \zeta)$ are generally understood as *off-shell backgrounds of string theory* and in this sense the Riemann-Hilbert approach realizes *an off-shell background independent formulation of string theory*.

In order to understand $g(t; \zeta)$ as off-shell backgrounds of string theory, it is worth mentioning the interpretation of *the position of cuts*. Taking into account the consideration given around Eq. (4.19), we can define the cuts on the off-shell background as a combination of general Stokes lines:

$$\text{Re}\left(g^{(i)}(t; \zeta) - g^{(j)}(t; \zeta)\right) = 0, \quad (5.16)$$

which is obtained by an analytic deformation of the matrix contour $\omega^{1/2}\mathcal{C}^{(k)}$ (so that it realizes the multi-cut boundary condition around $\zeta \rightarrow \infty$). Note that this consideration is possible after imposing proper Stokes phenomena which solve the multi-cut boundary condition, as it is carried out in Section 4.

This viewpoint also provides the following consideration: If one chooses $g(t; \zeta)$ as a semi-classical resolvent function $\tilde{\varphi}(t; \zeta)$, then the evaluation of Eqs. (5.11) and (5.10) in $g_{\text{str}} \rightarrow 0$,

$$\Psi_{\text{RH}}(t, \zeta) = Z(t; \zeta) e^{g(t; \zeta)} = \left[I_k + \cdots \right] e^{g(t; \zeta)}, \quad (5.17)$$

is a calculation of quantum corrections from the background spectral curve $g(t; \zeta)$ which is given by the semi-classical resolvent. Therefore, *if the resolvent background is a stable vacuum of this system, the non-perturbative corrections should be exponentially small*. This is the additional constraint for the Stokes multipliers and is referred to as *small-instanton condition*.

5.1 The small-instanton condition for the 2-cut critical point

Here we consider the small-instanton condition in the 2-cut $(1, 2)$ critical point. Mathematically, the Riemann-Hilbert problem in this case has been evaluated in [85, 86, 92–95] in the larger classes of Stokes multipliers (See the review [91]). In particular, according to the Deift-Zhou procedure [86], one first deforms the discontinuity lines \mathcal{K} to *anti-Stokes lines*. The concept of anti-Stokes lines depends on saddle points of the string background $g(t; \zeta)$:

$$\text{saddle points } \zeta_* = \zeta_{i,j}^{(n)}: \quad \left. \frac{\partial}{\partial \zeta} \left[g^{(i)}(t; \zeta) - g^{(j)}(t; \zeta) \right] \right|_{\zeta=\zeta_*} = 0, \quad (i, j = 1, 2, \dots, k). \quad (5.18)$$

Definition 11 (Anti-Stokes lines) *Anti-Stokes lines $\text{ASL}_{i,j}^{(n)}$ are defined for each pair of (i, j) as*

$$\text{ASL}_{i,j}^{(n)} = \left\{ \zeta \in \mathbb{C}; \text{Im} \left[g^{(i,j)}(t; \zeta) \right] = \text{Im} \left[g^{(i,j)}(t; \zeta_{i,j}^{(n)}) \right] \right\}, \quad (5.19)$$

where $\zeta_{i,j}^{(n)}$ is a saddle point of the function $g^{(i,j)}(t; \zeta) \equiv g^{(i)}(t; \zeta) - g^{(j)}(t; \zeta)$.

In the procedure of the Deift-Zhou method, one can choose the string background $g(t; \zeta)$, however, we know that the 2-cut $(1, 2)$ critical point has two phases with respect to the sign of t cosmological constant [42]. Therefore, we *choose* the string background according to the actual phase appearing in the two-cut matrix model:²⁹

$$g(t; \zeta) = \sigma_3 \left[\frac{1}{3} \zeta^3 + t\zeta + \dots \right] = \begin{cases} \sigma_3 \left[\frac{1}{3} (\zeta^2 + 2t)^{3/2} \right] & : \text{two-cut phase } (t > 0) \\ \sigma_3 \left[\frac{1}{3} \zeta^3 + t\zeta \right] & : \text{one-cut phase } (t < 0) \end{cases}. \quad (5.20)$$

Since we know that these curves are realized in the critical point as its stable vacua, these perturbative vacua should satisfy the small-instanton condition. Below we consider each case separately. We skip the calculation which is the same as that in [91].

The two-cut phase ($t > 0$) There are three saddle points of the function $g^{(1,2)}(t; \zeta) \equiv g^{(1)}(t; \zeta) - g^{(2)}(t; \zeta)$:

$$\zeta = \zeta_{1,2}^{(n)}: \quad \zeta_{1,2}^{(0)} = 0, \quad \zeta_{1,2}^{(\pm 1)} = \pm i\sqrt{2t}, \quad (5.21)$$

and the values of the function at these saddle points are

$$g^{(1,2)}(t; \zeta_{1,2}^{(0)}) = \frac{2}{3}(2t)^{3/2}, \quad g^{(1,2)}(t; \zeta_{1,2}^{(\pm 1)}) = 0. \quad (5.22)$$

Note the saddle-point value of the function $g^{(2,1)}(t; \zeta) = -g^{(1,2)}(t; \zeta)$. They are understood as instanton actions for the saddle points. The deformation of discrete lines \mathcal{K} to the DZ curves is given in Fig. 10.

²⁹Note that we are here imposing a *physical requirement*, by taking into account the Deift-Zhou method [86]. In the Deift-Zhou procedure, one considers an arbitrary Stokes multipliers, and the function $g(t; \zeta)$ is a function which we choose so that there is no divergence in the RH calculation. In this way, we can obtain the asymptotic form in t for these arbitrary Stokes multipliers. In this section, on the other hand, we impose a physical constraint in which the physical background $g(t; \zeta)$ obtained from the matrix models is stable perturbative background with small non-perturbative effects. Therefore, this constraint picks up the special and physical Stokes multipliers.

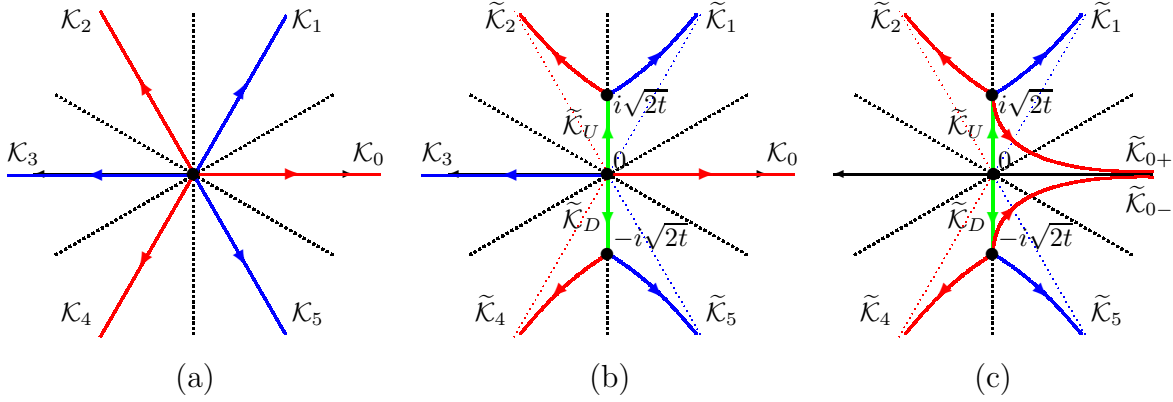


Figure 10: The discontinuity lines and the DZ curves in the two-cut (1, 2) critical point of the two-cut phase. a) The discontinuity lines \mathcal{K} . There are two kinds of lines: the one kind is the lines \mathcal{K}_{2n+1} on which the integral (5.15) only includes the contributions from the exponent $e^{g^{(1,2)}(\zeta)}$. The other kind is the lines \mathcal{K}_{2n} on which the integral (5.15) only includes the contributions from the exponent $e^{g^{(2,1)}(\zeta)}$. b) The DZ curves which are obtained from analytic deformation of the original lines \mathcal{K} . A large D-instanton effect appears around the origin on the line \mathcal{K}_3 . Therefore, we require $\alpha = 0$ so that this large instanton vanishes. c) The resulting DZ lines with $\alpha = 0$. Two lines along the real axes $\tilde{\mathcal{K}}_{0\pm}$ come from the Stokes matrices on the lines $\tilde{\mathcal{K}}_U$ and $\tilde{\mathcal{K}}_D$. Saddle point approximation on each line gives ZZ branes in the Liouville theory, however contributions from these lines are the same and canceled by the \mathbb{Z}_2 symmetry.

On the DZ curves, we then evaluate the integral (5.15) at saddle points [91]. The small-instanton condition becomes relevant when the saddle point $\zeta_{1,2}^{(0)} = 0$ of $g^{(1,2)}(t; \zeta)$ contributes in the Riemann-Hilbert integral (5.15). This happens in the integral on the curve \mathcal{K}_3 . The relevant part is given as

$$Z(t; \zeta) = \alpha E_{1,2} \int_{\mathcal{K}_3} \frac{d\lambda}{2\pi i} \frac{e^{g^{(1,2)}(t; \zeta)}}{\lambda - \zeta} + \dots \quad (5.23)$$

The parameter α is the Stokes multipliers of this system (2.59). Therefore, the small-instanton condition requires

$$\alpha = s_0 = s_3 = 0, \quad (5.24)$$

otherwise this perturbative vacuum (5.20) breaks down (or decays into some stable vacuum) by the large non-perturbative effects. Consequently, the solutions to the non-perturbative completion are finally fixed to be

$$\alpha = 0, \quad \beta = \pm 1 = -\gamma, \quad (5.25)$$

which is known as the Hastings-McLeod solution in the Painlevé II equation [89]. As it has been calculated in [89], the final result is given as³⁰

$$f(t) = -2\beta\sqrt{2t} + \dots \quad \text{with } \beta = \pm 1, \quad (5.26)$$

³⁰In this calculation, we use the local Riemann-Hilbert problems. Since the evaluation of the Riemann-Hilbert problem is not our purpose, we here skip the calculation. See the review [91]. An intuitive reason for vanishing the D-instanton effects (or physical interpretation of the mathematical result) is cancellation due to the \mathbb{Z}_2 symmetry of the system. For example, if one introduces the formal monodromy (as mentioned around (2.64), i.e. adding D0-brane charges in the background) then the instanton effect from the origin $\zeta = 0$ appears.

especially, the instanton effect which comes from a single ZZ-brane at the origin $\zeta = 0$ vanishes in this phase.

The one-cut phase ($t < 0$) There are two saddle points of the function $g^{(1,2)}(t; \zeta) \equiv g^{(1)}(t; \zeta) - g^{(2)}(t; \zeta)$:

$$\zeta = \zeta_{1,2}^{(n)} : \quad \zeta_{1,2}^{(\pm 1)} = \pm \sqrt{-t}, \quad (5.27)$$

and the values of the function at these saddle points are

$$g^{(1,2)}(t; \zeta_{1,2}^{(\pm 1)}) = \mp \frac{4}{3}(-t)^{3/2}. \quad (5.28)$$

The deformation of discrete lines \mathcal{K} to the DZ curves is given in Fig. 11. Note that existence of this phase also requires the same constraint $\alpha = 0$. By taking into account the solution to the non-perturbative completion (5.25), the Riemann-Hilbert integral (5.15) becomes the following simple contour integrals:

$$\begin{aligned} Z(t; \zeta) &= I_k + \beta E_{1,2} \int_{\mathcal{K}_{1,2}} \frac{d\lambda}{2\pi i} \frac{e^{g^{(1,2)}(t; \lambda)}}{\lambda - \zeta} - \beta E_{2,1} \int_{\mathcal{K}_{2,1}} \frac{d\lambda}{2\pi i} \frac{e^{g^{(2,1)}(t; \lambda)}}{\lambda - \zeta} + \dots, \\ &= I_k + \frac{\beta}{2\pi i} \left[i \sqrt{\frac{\pi}{2\sqrt{-t}}} \frac{E_{1,2}}{\sqrt{-t} - \zeta} - i \sqrt{\frac{\pi}{2\sqrt{-t}}} \frac{E_{2,1}}{-\sqrt{-t} - \zeta} \right] e^{-\frac{4}{3}(-t)^{3/2}} + \dots, \end{aligned} \quad (5.29)$$

therefore the asymptotic expression of $f(t)$ is given as

$$f(t) = -\frac{\beta}{\sqrt{2\pi\sqrt{-t}}} e^{-\frac{4}{3}(-t)^{3/2}} + \dots \quad \text{with } \beta = \pm 1. \quad (5.30)$$

See Eq. (2.48). It is worth mentioning that a similar expression was found in the 2-cut $(1, 2)$ critical points [71] which comes from an explicit expression of fermion state within the free-fermion formulation [25, 31, 32, 69], although the expression there is given by an infinite sum of super-matrix integrals.

5.2 The small-instanton condition for the k -cut critical points

Here we consider the small-instanton constraint in the k -cut $(1, 1)$ critical points. Since we here focus on the additional constraint, we only study the saddle point actions for the semi-classical string background and evaluation of the Riemann-Hilbert integrals is remained for future investigation. The classical backgrounds in these cases are calculated in [46] and given in terms of parameter z as

$$\begin{aligned} g(t; \zeta) &= \text{diag}(g^{(1)}(t; \zeta), \dots, g^{(k)}(t; \zeta)), \quad g^{(j)}(t; \zeta) = \int^{\omega^{-(j-1)}\zeta} y(z) dx(z), \\ \text{with } x(z) &= t \sqrt[k]{(z-c)^l (z-b)^{k-l}}, \quad y(z) = t \sqrt[k]{(z-c)^{k-l} (z-b)^l}, \end{aligned} \quad (5.31)$$

with $0 = cl + b(k-l)$. The index $l = (0, 1, 2, \dots, k-1)$ labels generally different solutions. The classical background $g(t; \zeta)$ is then expressed as

$$g^{(j)}(t; \zeta) = g^{(1)}(t; \omega^{-(j-1)}\zeta), \quad g^{(1)}(t; x) = \frac{1}{2}(z(x))^2 - (c+b)z(x). \quad (5.32)$$

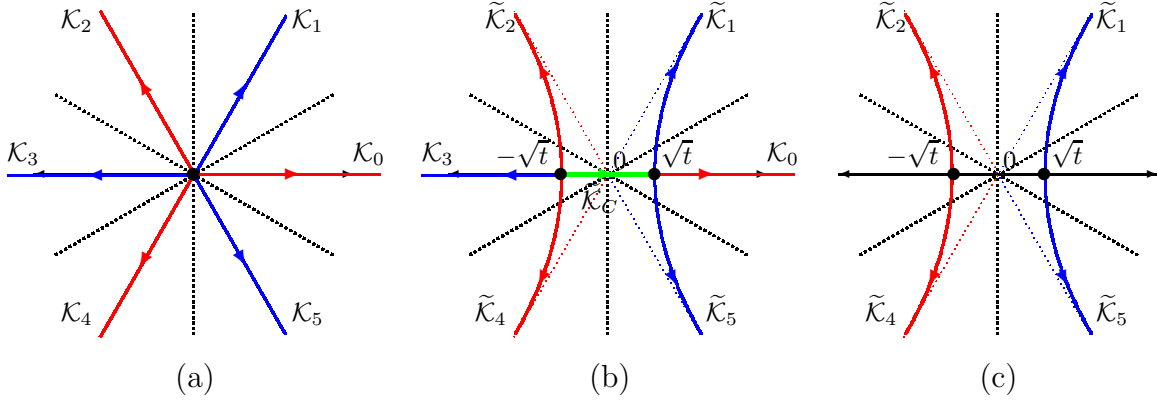


Figure 11: The discontinuity lines and the DZ curves in the two-cut (1,2) critical point of the one-cut phase. a) The discontinuity lines \mathcal{K} which is the same as two-cut phase. b) The DZ curves which are obtained from analytic deformation of the original lines \mathcal{K} . A large D-instanton effect appears around the saddle point $\zeta = +\sqrt{t}$ on the line \mathcal{K}_0 , and around the saddle point $\zeta = -\sqrt{t}$ on the line \mathcal{K}_3 . Therefore, we require $\alpha = 0$ so that these large instantons vanishes. c) The resulting DZ lines with $\alpha = 0$. By taking into account the sign of the Stokes multipliers, one observes that the integral (5.15) along connected lines $\tilde{\mathcal{K}}_2$ and $\tilde{\mathcal{K}}_4$ (and also $\tilde{\mathcal{K}}_1$ and $\tilde{\mathcal{K}}_5$ in the same way) can be considered as an integral on the single contour. Saddle point approximation on each line gives ZZ branes in the Liouville theory of the one-cut phase.

Here $z(x)$ is the inverse of the function $x(z)$ in Eq. (5.31). The saddle points for $g^{(i,j)}(t; \zeta) = g^{(i)}(t; \zeta) - g^{(j)}(t; \zeta)$ are given as

$$\frac{d}{d\zeta} g^{(i,j)}(t; \zeta) = 0 \quad \Leftrightarrow \quad \omega^{i-1} x(z) = \omega^{j-1} x(z'), \quad \omega^{-(i-1)} y(z) = \omega^{-(j-1)} y(z'), \quad (5.33)$$

and then this can be solved as

$$z' = z_{i,j}^{(n)} \equiv \left(\frac{b e^{\frac{i}{2}\chi_{i,j}^{(n)}} + c e^{-\frac{i}{2}\chi_{i,j}^{(n)}}}{2 \cos(\chi_{i,j}^{(n)}/2)} \right), \quad z = z_{j,i}^{(-n)} \equiv \left(\frac{b e^{-\frac{i}{2}\chi_{i,j}^{(n)}} + c e^{\frac{i}{2}\chi_{i,j}^{(n)}}}{2 \cos(\chi_{i,j}^{(n)}/2)} \right), \quad (5.34)$$

with $\chi_{i,j}^{(n)} \equiv 2\pi \frac{(i-j)+nk}{k-2l}$, ($n = 1, 2, \dots$). Substituting these values in Eq. (5.32), we obtain the saddle point action:

$$\begin{aligned} g^{(i,j)}(t; \zeta_{i,j}^{(n)}) &= \frac{1}{2} \left((z_{i,j}^{(n)})^2 - (z_{j,i}^{(-n)})^2 \right) - (b+c) \left(z_{i,j}^{(n)} - z_{j,i}^{(-n)} \right) \\ &= i \frac{c^2 - b^2}{2} \tan\left(\frac{\chi_{i,j}^{(n)}}{2}\right) \in i\mathbb{R}. \end{aligned} \quad (5.35)$$

This means that the saddle point action always contributes order $\mathcal{O}(g_{\text{str}}^0)$ and then identified as perturbative corrections (not as instantons). Therefore, there is no additional (small-instanton) constraints on the solutions obtained in Section 4.

6 Conclusion and discussions

In this paper, we give concrete solutions to the non-perturbative completion in the k -cut two-matrix models by a quantitative study of Stokes phenomena. The non-perturbative

completion problem consists of the multi-cut boundary condition for the orthonormal polynomial systems and the non-perturbative stability condition for the semi-classical spectral curves in the large N limit. By carrying out these procedures, we demonstrated two classes of solutions, which are referred to as discrete and continuum solutions. Interestingly, the solutions possess kind of “charges” in terms of Young diagram representation.

We note that the continuum solutions to the non-perturbative completion still include continuous free parameters, although the two-cut cases have been completely fixed. It is conceivable that we might need to rely on further independent physical arguments to reduce these degrees of freedom, here we would like to interpret these free parameters as physical moduli parameters in the non-perturbative region of the string theory. Since the strong-coupling dual theory of the multi-cut matrix models seems to be non-critical M theory [44], these continuous parameters would correspond to the non-perturbative (non-normalizable) moduli space of M theory, $\mathcal{M}_{\text{M-theory}}^{(\text{non-norm.})}$ which is a distinct parameter space from the string-theory moduli space, $\mathcal{M}_{\text{string}}^{(\text{non-norm.})}$ and $\mathcal{M}_{\text{string}}^{(\text{norm.})}$. Below we provide a list of issues which deserve further exploration.

- In this paper, we have solved Stokes phenomena in \mathbb{Z}_k -symmetric critical points. It is also interesting to consider similar program in the fractional-superstring critical points [43]. In particular, we would like to see the emergence of the non-critical M theory from the $k \rightarrow \infty$ limit [44].
- Our procedure is directly related to Riemann-Hilbert calculus. It is useful to examine higher order instanton sectors and generalize the results in [78].
- In this paper, we focus on the cases with $\hat{p} = 1$ and small \hat{q} . In order to extend this procedure to the general \hat{q} cases, one should resolve several complexities as shown in Eq. (4.40). It is of great interest to obtain the Stokes multipliers in higher (\hat{p}, \hat{q}) critical points. In particular, evaluation in the bosonic cases would clarify the issue raised in [61]. Also we have to take into account the smoothing of the cuts as shown in [52] (also see Appendix A).
- It is interesting to investigate *whether the Riemann-Hilbert representation can be written in language of matrix models?* This resembles the supermatrix models [71] which appear by evaluating tau-function in terms of free fermions. Also it is interesting to compare it with Kontsevich type matrix models [105] and also with the non-perturbative topological string-theory block recently proposed in [106].
- The Riemann-Hilbert representation is a background independent formulation, which allows us to introduce general off-shell background in string theory. Therefore, it is interesting to study physics in off-shell backgrounds and general concept of background independence in matrix models/string theory.
- In the multi-cut matrix models, there are two kinds of perturbative string vacua [44]: One is perturbatively isolated sectors (perturbative superselection sectors) which are decoupled with other sectors in all-order perturbation theory. This phenomenon is an origin of the extra-dimension in M theory. The other is perturbative vacua in the string-theory moduli space. For survey for the second vacua, the Riemann-Hilbert representation is even more powerful, since the off-shell moduli space is

understand as the space of off-shell string-theory backgrounds. Furthermore, the \mathbb{Z}_k symmetric critical points in the multi-cut matrix models have several perturbative vacua which satisfy loop equations. Therefore, it is interesting to study non-perturbative string-theory landscape from the Riemann-Hilbert approach. In particular, it might be possible to identify which observables are suitable for a description of *a potential picture in the moduli space*.

- We obtained several solutions to Stokes phenomena which are characterized by several charges carried by Young diagrams. What is the physical meaning of our solutions? Any relation to W-symmetry or WZNW?
- Our solutions are natural generalizations of the Hastings-McLeod solution in the Painlevé II equation. The Hastings-McLeod solution is known to have several special features, for instance analyticity of the solution (See also [91]). Therefore, it is mathematically interesting to understand the analyticity of the solutions in t and to identify the standing point of our solutions in general solutions of the string equations.
- As is well-known, the integrable deformations in the usual integrable system correspond to the moduli space of worldsheet conformal field theory. On the other hand, non-trivial deformations of our solutions can be interpreted as non-perturbative integrable deformations in *physical solutions of string equations*. Therefore, these deformations are related to the moduli space of the dynamical degree of freedom in the strong coupling region, i.e. degree of freedom in non-critical M theory. It is interesting if there is a comprehensive understanding of these non-perturbative integrable deformations.

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A Stokes phenomenon in the Airy function

The non-perturbative relations between the resolvent and the orthonormal polynomials are first studied in [52] in the $(2, 1)$ critical point of bosonic minimal string. Since this study also uncovers another aspect of cuts in the resolvent curves for the cases of $\hat{p} \geq 2$, we here briefly review the results and summarize the key points.

In the bosonic $(2, 1)$ critical point, the orthonormal polynomials satisfy the following differential equation:

$$\zeta \Psi_{\text{orth}}(t; \zeta) = (\partial^2 + u(t)) \Psi_{\text{orth}}(t; \zeta), \quad (\text{A.1})$$

$$g_{\text{str}} \frac{\partial}{\partial \zeta} \Psi_{\text{orth}}(t; \zeta) = \partial \Psi_{\text{orth}}(t; \zeta). \quad (\text{A.2})$$

By taking into account the definition $\partial \equiv g_{\text{str}} \partial_t$, one can show that the orthonormal polynomial is given as Airy function:

$$0 = \left(g_{\text{str}}^2 \frac{\partial^2}{\partial \zeta^2} - \zeta - t \right) \Psi_{\text{orth}}(t; \zeta), \quad \Psi_{\text{orth}}(t; \zeta) = \text{Ai}(\zeta + t). \quad (\text{A.3})$$

Here we have concluded $u(t) = -t$ by imposing the integrability condition of (A.1) and (A.2), and also have chosen the damping solution (Airy function) as the physical solution [52]:

$$\Psi_{\text{orth}}(t; \zeta) \rightarrow 0, \quad \zeta \rightarrow \infty. \quad (\text{A.4})$$

As it is well-known, the asymptotic behavior of the orthonormal polynomial $\Psi_{\text{orth}}(t; \zeta)$ (i.e. the Airy function) around the real axes, $\zeta \rightarrow \pm\infty$, is given as

$$\Psi_{\text{orth}}(t; \zeta) \underset{\text{asym}}{\simeq} \left(\frac{g_{\text{str}} \pi}{(\zeta + t)^{1/2}} \right)^{1/2} e^{-\frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}} + \dots, \quad (\text{A.5})$$

in $\zeta \rightarrow \infty$ with the angle, $|\arg(\zeta)| < \pi$, and

$$\Psi_{\text{orth}}(t; \zeta) \underset{\text{asym}}{\simeq} \left(\frac{g_{\text{str}} \pi}{(\zeta + t)^{1/2}} \right)^{1/2} \left[e^{-\frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}} + i e^{\frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}} \right] + \dots, \quad (\text{A.6})$$

in $\zeta \rightarrow e^{\pi i} \times \infty$ with the angle, $|\arg(-\zeta)| < 2\pi/3$. Note that both two expressions in the intersections, $\pi/3 < |\arg(\zeta)| < \pi$, have common asymptotic expansions, and therefore, the appearance/disappearance of different exponents in different asymptotic regions is understood as the Stokes phenomenon. As a consequence, the resolvent in the weak coupling limit $g_{\text{str}} \rightarrow 0$ is smooth in ζ with $\arg(\zeta) < \pi$, and the discontinuity only appears along $\zeta \in (-\infty, -t)$, that is,

$$\lim_{\epsilon \rightarrow \pm 0} \left[\lim_{g_{\text{str}} \rightarrow 0} \Psi_{\text{orth}}(t; \zeta + i\epsilon) \right] \sim e^{\mp \frac{2}{3g_{\text{str}}}(\zeta+t)^{3/2}}, \quad \zeta \in (-\infty, -t). \quad (\text{A.7})$$

An important point in [52] is that the resolvent curve itself has a cut around $\zeta \rightarrow \infty$. However the explicit cuts are smeared by the superposition of the exponents $e^{(\zeta+t)^{3/2}}$ and $e^{-(\zeta+t)^{3/2}}$. Note that the solution corresponding to matrix models can be chosen by the single condition Eq. (A.4). This is due to the simplicity of Airy system. In more general system, however, one needs the multi-cut boundary condition proposed in this paper.

B Lax operators in the multi-cut matrix models

Here we summarize the Lax operators used in this paper.

B.1 The \mathbb{Z}_k symmetric $(1, 1)$ critical points

This class of critical points are characterized by the following Lax operators:

$$\begin{aligned} \mathbf{P}(t; \partial) &= \Gamma \partial + H(t), \\ \mathbf{Q}(t; \partial) &= (\Gamma^{-2}(t; \partial) \mathbf{P}(t; \partial))_+ - \mu (\Gamma^{-1}(t; \partial))_+ \\ &= \Gamma^{-1} \partial - \Gamma^{-1} H \Gamma^{-1} - \mu \Gamma^{-1}. \end{aligned} \quad (\text{B.1})$$

Note that the \mathbb{Z}_k symmetry requires

$$H(t) = \begin{pmatrix} 0 & * & & & \\ & 0 & * & & \\ & & \ddots & \ddots & \\ & & & 0 & * \\ * & & & & 0 \end{pmatrix}, \quad (\text{B.2})$$

and the Lax operator $\mathbf{\Gamma}(t; \partial)$ is defined as

$$\mathbf{\Gamma}(t; \partial) = \Gamma + \sum_{n=1}^{\infty} S_n(t) \partial^{-n}, \quad (\mathbf{\Gamma}(t; \partial))^k = I_k, \quad [\mathbf{\Gamma}(t; \partial), \mathbf{P}(t; \partial)] = 0. \quad (\text{B.3})$$

From these operators, one can calculate the operator $\mathcal{Q}(t; \zeta)$ (see Eq. (2.14)) which is given as

$$\mathcal{Q}(t; \zeta) = \Gamma^{-2} \zeta - \Gamma^{-1} (\{\Gamma^{-1}, H\} + \mu). \quad (\text{B.4})$$

The coefficients of the asymptotic expansion (2.18) are then calculated as

$$\begin{aligned} \varphi(t; \zeta) &= \frac{(\Gamma^{-1} \zeta)^2}{2} - \mu \Gamma^{-1} \zeta + \mathcal{O}(1/\zeta), \\ Y(t; \zeta) &= I_k + \frac{1}{\zeta} \text{adj}^{-1}(\Gamma^{-2}) [\Gamma^{-1} \{\Gamma^{-1}, H(t)\}] + \mathcal{O}(1/\zeta^2), \end{aligned} \quad (\text{B.5})$$

where adj^{-1} is the inverse operator of $\text{adj}(A)[B] = AB - BA$. In the $k = 3$ case, by using the formula, $\text{adj}^{-1}(\Gamma^{-1})[X] = [\Gamma^{-1}, X]/3$, one can show

$$Y_1(t) = \frac{1}{3} (H(t) - \Gamma^{-1} H(t) \Gamma). \quad (\text{B.6})$$

Here we have checked that $\varphi_0(t) = 0$ is true for first few cases $k = 3, 4, 5$ and this is consistent with our solutions.

B.2 Fractional-superstring $(\hat{p}, \hat{q}) = (1, 2)$ critical points ($r = 3$)

In this case, we only study $k = 2$ case, but generally one can calculate as follows: The Lax operators in these cases are

$$\begin{aligned} \mathbf{P}(t; \partial) &= \Gamma \partial + H(t), \quad \mathbf{Q}(t; \partial) = (\Gamma^{-1}(t; \partial) \mathbf{P}^2(t; \partial))_+ - \mu (\Gamma^{-1}(t; \partial))_+ \\ &= \Gamma \partial^2 + H(t) \partial - S_2(t) - \mu \Gamma^{-1} \end{aligned} \quad (\text{B.7})$$

Therefore, the operator $\mathcal{Q}(t; \zeta)$ is given as

$$\mathcal{Q}(t; \zeta) = \Gamma^{-1} \zeta^2 - \Gamma^{-1} H(t) \zeta - \partial H(t) - S_2(t) - \mu \Gamma^{-1}, \quad (\text{B.8})$$

or

$$\mathcal{Q}_{-3}(t) = \Gamma^{-1}, \quad \mathcal{Q}_{-2}(t) = -\Gamma^{-1} H(t), \quad \mathcal{Q}_{-1}(t) = -\partial H(t) - S_2(t) - \mu \Gamma^{-1}. \quad (\text{B.9})$$

Here $S_2(t)$ satisfies³¹

$$[\Gamma, S_2(t)] + \Gamma \partial H = 0, \quad \{\Gamma, \dots, \Gamma, S_2(t)\}_k + \{\Gamma, \dots, \Gamma, H(t), H(t)\}_k = 0. \quad (\text{B.11})$$

In the $k = 2$ case, $S_2(t)$ is given as

$$S_2(t) = \frac{1}{2} \left(\sigma_1 f^2(t) - i \sigma_2 \partial f(t) \right), \quad H(t) = i \sigma_2 f(t). \quad (\text{B.12})$$

The coefficients of the asymptotic expansion are given as

$$\begin{aligned} \varphi(\zeta) &= \sigma_1 \left(\frac{\zeta^3}{3} - \mu \zeta \right) + \mathcal{O}(1/\zeta), \\ Y(\zeta) &= I_2 + \frac{1}{2} i \sigma_2 \frac{f(t)}{\zeta} - \frac{1}{4} \sigma_3 \frac{\partial f(t)}{\zeta^2} + \frac{1}{8} i \sigma_2 \frac{f^3(t) - 4\mu f(t)}{\zeta^3} + \mathcal{O}(1/\zeta^4). \end{aligned} \quad (\text{B.13})$$

C Supplements to Theorem 5 and Theorem 6

In this appendix, we first show the explicit form of the linear expressions Eq. (4.38) and Eq. (4.48), and then show some examples. Before we focus on these cases, we summarize the general properties of these systems by introducing the following four categories of the indices i of $y_{n,i}$:

$$\begin{aligned} \text{(I)} \quad & 1 \leq i \leq \left\lfloor \frac{k+3}{4} \right\rfloor =: A, & \text{(II)} \quad & B := \left\lfloor \frac{k+3}{4} \right\rfloor + 1 \leq i \leq \frac{k+1}{2}, \\ \text{(III)} \quad & \frac{k+1}{2} + 1 \leq i \leq \left\lfloor \frac{3k+3}{4} \right\rfloor =: C, & \text{(IV)} \quad & D := \left\lfloor \frac{3k+3}{4} \right\rfloor + 1 \leq i \leq k, \end{aligned} \quad (\text{C.1})$$

which is closely related to the multi-cut boundary conditions Eq. (4.33) and Eq. (4.45). On the other hand, we show this division in the profile $\mathcal{J}_{k,2}^{(\text{sym})}$. Here we show the categories

³¹We define the symmetric product $\{A_1, A_2, \dots, A_k\}_k$ as

$$\left(\sum_i a_i \right)^k \equiv \sum_{i_1, i_2, \dots, i_k} \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}_k. \quad (\text{B.10})$$

of I and III with *italic font* and the categories of II and IV with **bold font**:

$$\underline{k = 4k_0 + 1, (k_0 \in \mathbb{N}) :}$$

$$\mathcal{J}_{k,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{B} & (\mathbf{D} & \mathbf{D} + 1) & (A & \mathbf{B} + 1) & (C & \mathbf{D} + 2) & (A - 1 & \mathbf{B} + 2) & \dots \\ \hline (\mathbf{D} & \mathbf{B}) & (A & \mathbf{D} + 1) & (C & \mathbf{B} + 1) & (A - 1 & \mathbf{D} + 2) & (C - 1 & \dots \\ \hline \mathbf{D} & (A & \mathbf{B}) & (C & \mathbf{D} + 1) & (A - 1 & \mathbf{B} + 1) & (C - 1 & \mathbf{D} + 2) & \dots \\ \hline (A & \mathbf{D}) & (C & \mathbf{B}) & (A - 1 & \mathbf{D} + 1) & (C - 1 & \mathbf{B} + 1) & (A - 2 & \dots \\ \hline \dots & (4 & \frac{k-1}{2}) & (\frac{k+7}{2} & \mathbf{k}) & (3 & \frac{k+1}{2}) & (\frac{k+5}{2} & 1) & (2 & \frac{k+3}{2}) \\ \hline \dots & \mathbf{k} - 1) & (\frac{k+7}{2} & \frac{k-1}{2}) & (3 & \mathbf{k}) & (\frac{k+5}{2} & \frac{k+1}{2}) & (2 & 1) & \frac{k+3}{2} \\ \hline \dots & (\frac{k+7}{2} & \mathbf{k} - 1) & (3 & \frac{k-1}{2}) & (\frac{k+5}{2} & \mathbf{k}) & (2 & \frac{k+1}{2}) & (\frac{k+3}{2} & 1) \\ \hline \dots & \frac{k-3}{2}) & (3 & \mathbf{k} - 1) & (\frac{k+5}{2} & \frac{k-1}{2}) & (2 & \mathbf{k}) & (\frac{k+3}{2} & \frac{k+1}{2}) & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array} \quad (\text{C.2})$$

and some concrete examples ($k = 9$ and 13) are

$$\mathcal{J}_{9,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{4} & (\mathbf{8} & \mathbf{9}) & (3 & \mathbf{5}) & (7 & 1) & (2 & 6) \\ \hline (\mathbf{8} & \mathbf{4}) & (3 & \mathbf{9}) & (7 & \mathbf{5}) & (2 & 1) & 6 \\ \hline \mathbf{8} & (3 & \mathbf{4}) & (7 & \mathbf{9}) & (2 & \mathbf{5}) & (6 & 1) \\ \hline (3 & \mathbf{8}) & (7 & \mathbf{4}) & (2 & \mathbf{9}) & (6 & \mathbf{5}) & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array},$$

$$\mathcal{J}_{13,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{5} & (\mathbf{11} & \mathbf{12}) & (4 & \mathbf{6}) & (10 & \mathbf{13}) & (3 & \mathbf{7}) & (9 & 1) & (2 & 8) \\ \hline (\mathbf{11} & \mathbf{5}) & (4 & \mathbf{12}) & (10 & \mathbf{6}) & (3 & \mathbf{13}) & (9 & \mathbf{7}) & (2 & 1) & 8 \\ \hline \mathbf{11} & (4 & \mathbf{5}) & (10 & \mathbf{12}) & (3 & \mathbf{6}) & (9 & \mathbf{13}) & (2 & \mathbf{7}) & (8 & 1) \\ \hline (4 & \mathbf{11}) & (10 & \mathbf{5}) & (3 & \mathbf{12}) & (9 & \mathbf{6}) & (2 & \mathbf{13}) & (8 & \mathbf{7}) & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array}. \quad (\text{C.3})$$

$$\underline{k = 4k_0 + 3, (k_0 \in \mathbb{N}) :}$$

$$\mathcal{J}_{k,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{D} & (\mathbf{B} & \mathbf{B} + 1) & (C & \mathbf{D} + 1) & (A & \mathbf{B} + 2) & (C - 1 & \mathbf{D} + 2) & \dots \\ \hline (\mathbf{B} & \mathbf{D}) & (C & \mathbf{B} + 1) & (A & \mathbf{D} + 1) & (C - 1 & \mathbf{B} + 2) & (A - 1 & \dots \\ \hline \mathbf{B} & (C & \mathbf{D}) & (A & \mathbf{B} + 1) & (C - 1 & \mathbf{D} + 1) & (A - 1 & \mathbf{B} + 2) & \dots \\ \hline (C & \mathbf{B}) & (A & \mathbf{D}) & (C - 1 & \mathbf{B} + 1) & (A - 1 & \mathbf{D} + 1) & (C - 2 & \dots \\ \hline \dots & (4 & \frac{k-1}{2}) & (\frac{k+7}{2} & \mathbf{k}) & (3 & \frac{k+1}{2}) & (\frac{k+5}{2} & 1) & (2 & \frac{k+3}{2}) \\ \hline \dots & \mathbf{k} - 1) & (\frac{k+7}{2} & \frac{k-1}{2}) & (3 & \mathbf{k}) & (\frac{k+5}{2} & \frac{k+1}{2}) & (2 & 1) & \frac{k+3}{2} \\ \hline \dots & (\frac{k+7}{2} & \mathbf{k} - 1) & (3 & \frac{k-1}{2}) & (\frac{k+5}{2} & \mathbf{k}) & (2 & \frac{k+1}{2}) & (\frac{k+3}{2} & 1) \\ \hline \dots & \frac{k-3}{2}) & (3 & \mathbf{k} - 1) & (\frac{k+5}{2} & \frac{k-1}{2}) & (2 & \mathbf{k}) & (\frac{k+3}{2} & \frac{k+1}{2}) & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array}, \quad (\text{C.4})$$

and some concrete examples ($k = 7$ and 11) are

$$\mathcal{J}_{7,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{7} & (\mathbf{3} & \mathbf{4}) & (6 & 1) & (2 & 5) \\ \hline (\mathbf{3} & \mathbf{7}) & (6 & \mathbf{4}) & (2 & 1) & 5 \\ \hline \mathbf{3} & (6 & \mathbf{7}) & (2 & \mathbf{4}) & (5 & 1) \\ \hline (6 & \mathbf{3}) & (2 & \mathbf{7}) & (5 & \mathbf{4}) & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array},$$

$$\mathcal{J}_{11,2}^{(\text{sym})} = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c} \mathbf{10} & (4) & \mathbf{5} & (9) & \mathbf{11} & (3) & \mathbf{6} & (8) & (1) & (2) & (7) \\ \hline (4) & \mathbf{10} & (9) & \mathbf{5} & (3) & \mathbf{11} & (8) & \mathbf{6} & (2) & (1) & 7 \\ \hline 4 & (9) & \mathbf{10} & (3) & \mathbf{5} & (8) & \mathbf{11} & (2) & \mathbf{6} & (7) & (1) \\ \hline (9) & (4) & (3) & \mathbf{10} & (8) & \mathbf{5} & (2) & \mathbf{11} & (7) & \mathbf{6} & 1 \end{array} \right] : \begin{array}{l} J_3 \\ J_2 \\ J_1 \\ J_0 \end{array} . \quad (\text{C.5})$$

If one follows trajectories of numbers, One may notice that the trajectories of the numbers in the region I and III (written by *italic font*) almost form slash shape, “/”, and the trajectories of the numbers in the region II and IV (written by **bold font**) almost form backslash shape, “\”. In the left-hand and right-hand ends of the profile $\mathcal{J}_{k,2}^{(\text{sym})}$, there are a few exceptions which form curved shape as “<” and so on. From this property, we can see the following facts:

- Stokes multipliers $s_{l,i,j} \leftrightarrow (j|i)_l$ are almost given by $i \in \text{II, IV}$ (**bold font**) and $j \in \text{I, III}$ (*italic font*). We emphasize this fact by writing $s_{l,\mathbf{i},j} \leftrightarrow (j|\mathbf{i})_l$. Then there are only a few exceptions, $s_{*,\mathbf{i},\mathbf{j}}$ ($\mathbf{i}, \mathbf{j} \in \text{II, IV}$) and $s_{*,i,j}$ ($i, j \in \text{I, III}$), which appear in the left-hand and right-hand ends of the profile $\mathcal{J}_{k,2}^{(\text{sym})}$. Interestingly, there is no Stokes multipliers of the type $s_{l,i,\mathbf{j}}$ with $i \in \text{I, III}$ and $\mathbf{j} \in \text{II, IV}$ in this $r = 2$ case.
- From this fact, one can show that the relation between the symmetric Stokes multipliers $s_{0,i,j}^{(\text{sym})}$ and the fine Stokes multipliers $s_{l,i,j}$ (Eq. (3.16)) are almost trivial: $s_{0,i,j}^{(\text{sym})} = s_{\exists l,i,j}$ for almost all $(j|i)_l \in \mathcal{J}_{k,2}^{(\text{sym})}$, and that only the following few multipliers are the exceptions:

$$s_{0,1,2}^{(\text{sym})} = s_{2,1,2} + s_{1,1,\frac{k+3}{2}} s_{3,\frac{k+3}{2},2}, \quad s_{0,\frac{k+1}{2},2}^{(\text{sym})} = s_{1,\frac{k+1}{2},2} + s_{0,\frac{k+1}{2},\frac{k+3}{2}} s_{3,\frac{k+3}{2},2}, \quad (\text{C.6})$$

and

$$\begin{aligned} & \underline{k = 4k_0 + 1 :} \\ & s_{0,\mathbf{B},A}^{(\text{sym})} = s_{1,\mathbf{B},A} + s_{2,\mathbf{B},\mathbf{D}} s_{0,\mathbf{D},A}, \quad s_{0,\mathbf{D}+1,A}^{(\text{sym})} = s_{1,\mathbf{D}+1,A} + s_{3,\mathbf{D}+1,\mathbf{D}} s_{0,\mathbf{D},A}; \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} & \underline{k = 4k_0 + 3 :} \\ & s_{0,\mathbf{D},C}^{(\text{sym})} = s_{1,\mathbf{D},C} + s_{2,\mathbf{D},\mathbf{B}} s_{0,\mathbf{B},C}, \quad s_{0,\mathbf{B}+1,C}^{(\text{sym})} = s_{2,\mathbf{B}+1,C} + s_{3,\mathbf{B}+1,\mathbf{B}} s_{0,\mathbf{B},C}. \end{aligned} \quad (\text{C.8})$$

- These relations are important not only because they are used in deriving the results in this appendix, but also because they provide a concrete example which guarantees the claim shown in Eq. (3.37). In particular, we expect that one can extend this discussion to the general systems of $(k, r; \gamma)$ which are controlled by the method developed in Section 3.

C.1 The explicit form of Eq. (4.38) and Eq. (4.48)

Below is the explicit form of the linear expressions Eq. (4.38) and Eq. (4.48). Note the function $\epsilon(k)$ appears in these formulas is given as

$$\epsilon(k) = \begin{cases} 0 & (k = 4k_0 + 1) \\ 1 & (k = 4k_0 + 3) \end{cases} . \quad (\text{C.9})$$

The linear expression of Eq. (4.38)

Region I:

$$y_{n,i}(\{y_{m,1}\}_{m \in \mathbb{Z}}) = y_{n+i-1,1} \neq 0, \quad i \in (\text{I}), \quad (\text{C.10})$$

Region II:

$$\begin{aligned} y_{n,B+j}(\{y_{m,1}\}_{m \in \mathbb{Z}}) &\equiv y_{n+A+j,1} + \sum_{a=0}^{j-\epsilon(k)} s_{0,B+j-a,A-j+a+\epsilon(k)}^{(\text{sym})} \times y_{n+A-j-1+2a+\epsilon(k),1} + \\ &+ \sum_{a=0}^{j-1} s_{0,B+j-a,A-j+a+1+\epsilon(k)}^{(\text{sym})} \times y_{n+A-j+2a+\epsilon(k),1}, \end{aligned} \quad (\text{C.11})$$

Region III:

$$y_{n,i}(\{y_{m,1}\}_{m \in \mathbb{Z}}) = 0, \quad i \in (\text{III}). \quad (\text{C.12})$$

Region IV:

$$\begin{aligned} y_{n,D+j}(\{y_{m,1}\}_{m \in \mathbb{Z}}) &\equiv \sum_{a=0}^j s_{0,D+j-a,A-j+a}^{(\text{sym})} \times y_{n+A-j-1+2a,1} \\ &+ \sum_{a=0}^{j-1+\epsilon(k)} s_{0,D+j-a,A-j+a+1}^{(\text{sym})} \times y_{n+A-j+2a,1}. \end{aligned} \quad (\text{C.13})$$

The linear expression of Eq. (4.48)

Region I:

$$\tilde{y}_{n,i}(\{\tilde{y}_{m,\frac{k+3}{2}}\}_{m \in \mathbb{Z}}) = 0, \quad i \in (\text{I}), \quad (\text{C.14})$$

Region II:

$$\begin{aligned} \tilde{y}_{n,B+j}(\{\tilde{y}_{m,\frac{k+3}{2}}\}_{m \in \mathbb{Z}}) &\equiv \sum_{a=0}^j s_{0,B+j-a,C-j+a}^{(\text{sym})} \times \tilde{y}_{n+C-j-\frac{k+3}{2}+2a,\frac{k+3}{2}} + \\ &+ \sum_{a=0}^{j-\epsilon(k)} s_{0,B+j-a,C-j+a+1}^{(\text{sym})} \times \tilde{y}_{n+C-j+2a+1-\frac{k+3}{2},\frac{k+3}{2}}, \end{aligned} \quad (\text{C.15})$$

Region III:

$$\tilde{y}_{n,i}(\{\tilde{y}_{m,\frac{k+3}{2}}\}_{m \in \mathbb{Z}}) = \tilde{y}_{n+i-\frac{k+3}{2},\frac{k+3}{2}} \neq 0, \quad i \in (\text{III}), \quad (\text{C.16})$$

Region IV:

$$\begin{aligned} \tilde{y}_{n,D+j}(\{\tilde{y}_{m,\frac{k+3}{2}}\}_{m \in \mathbb{Z}}) &\equiv \tilde{y}_{n+D+j-\frac{k+3}{2},\frac{k+3}{2}} + \\ &+ \sum_{a=0}^{j-1} s_{0,D+j-a,C-j+a+1-\epsilon(k)}^{(\text{sym})} \times \tilde{y}_{n+C-j+a+1-\epsilon(k)-\frac{k+3}{2},\frac{k+3}{2}} + \\ &+ \sum_{a=0}^{j-1+\epsilon(k)} s_{0,D+j-a,C-j+a+2-\epsilon(k)}^{(\text{sym})} \times \tilde{y}_{n+C-j+a+2-\epsilon(k)-\frac{k+3}{2},\frac{k+3}{2}}. \end{aligned} \quad (\text{C.17})$$

C.2 Some examples of Theorem 5

Below we show the concrete expressions of the equations in Theorem 5 for some special cases ($k = 5, 7, 9$ and 11). First of all, the vectors are expressed only by using $\{y_{n,1}\}_{n \in \mathbb{Z}}$:

$$\begin{aligned}
\underline{k=5}: \quad Y^{(n)} &= \begin{pmatrix} y_{n,1} \\ y_{n+1,1} \\ -s_{0,4,2}^{(\text{sym})} y_{n,1} \\ 0 \\ s_{0,5,2}^{(\text{sym})} y_{n+1,1} \end{pmatrix}, \quad \underline{k=7}: \quad Y^{(n)} = \begin{pmatrix} y_{n,1} \\ y_{n+1,1} \\ y_{n+2,1} \\ -s_{0,5,2}^{(\text{sym})} y_{n,1} \\ 0 \\ 0 \\ s_{0,7,2}^{(\text{sym})} y_{n+1,1} + s_{0,7,3}^{(\text{sym})} y_{n+2,1} \end{pmatrix}, \\
\underline{k=9}: \quad Y^{(n)} &= \begin{pmatrix} y_{n,1} \\ y_{n+1,1} \\ y_{n+2,1} \\ y_{n+3,1} + s_{0,4,3}^{(\text{sym})} y_{n+2,1} \\ -s_{0,6,2}^{(\text{sym})} y_{n,1} \\ 0 \\ 0 \\ s_{0,8,3}^{(\text{sym})} y_{n+2,1} \\ s_{0,8,3}^{(\text{sym})} y_{n+3,1} + s_{0,9,3}^{(\text{sym})} y_{n+2,1} + s_{0,9,2}^{(\text{sym})} y_{n+1,1} \end{pmatrix}, \\
\underline{k=11}: \quad Y^{(n)} &= \begin{pmatrix} y_{n,1} \\ y_{n+1,1} \\ y_{n+2,1} \\ y_{n+3,1} \\ y_{n+4,1} + s_{0,5,3}^{(\text{sym})} y_{n+2,1} + s_{0,5,4}^{(\text{sym})} y_{n+3,1} \\ -s_{0,7,2}^{(\text{sym})} y_{n,1} \\ 0 \\ 0 \\ 0 \\ s_{0,10,3}^{(\text{sym})} y_{n+2,1} + s_{0,10,4}^{(\text{sym})} y_{n+3,1} \\ s_{0,10,3}^{(\text{sym})} y_{n+3,1} + s_{0,10,4}^{(\text{sym})} y_{n+4,1} + s_{0,11,2}^{(\text{sym})} y_{n+1,1} + s_{0,11,3}^{(\text{sym})} y_{n+2,1} \end{pmatrix}.
\end{aligned} \tag{C.18}$$

Secondly, the multi-cut BC recursions are expressed as

$$\underline{k=5}$$

$$\mathcal{F}_5[y_{n,1}] \equiv y_{n+2,1} + s_{1,3,2} y_{n+1,1} + s_{3,4,2} y_{n,1} = 0,$$

$$\mathcal{G}_5[y_{n,1}] \equiv s_{0,5,2} y_{n+2,1} + s_{2,1,2} y_{n+1,1} - y_{n,1} = 0,$$

$$\underline{k=7}$$

$$\mathcal{F}_7[y_{n,1}] \equiv y_{n+3,1} + s_{3,4,3} y_{n+2,1} + s_{1,4,2} y_{n+1,1} + s_{3,5,2} y_{n,1} = 0,$$

$$\mathcal{G}_7[y_{n,1}] \equiv s_{2,7,3} y_{n+3,1} + s_{0,7,2} y_{n+2,1} + s_{2,1,2} y_{n+1,1} - y_{n,1} = 0,$$

$$\underline{k=9}$$

$$\mathcal{F}_9[y_{n,1}] \equiv y_{n+4,1} + s_{1,4,3} y_{n+3,1} + s_{3,5,3} y_{n+2,1} + s_{1,5,2} y_{n+1,1} + s_{3,6,2} y_{n,1} = 0,$$

$$\mathcal{G}_9[y_{n,1}] \equiv s_{0,8,3} y_{n+4,1} + s_{2,9,3} y_{n+3,1} + s_{0,9,2} y_{n+2,1} + s_{2,1,2} y_{n+1,1} - y_{n,1} = 0,$$

$$k = 11$$

$$\begin{aligned}\mathcal{F}_{11}[y_{n,1}] &\equiv y_{n+5,1} + s_{3,5,4} y_{n+4,1} + s_{1,5,3} y_{n+3,1} + s_{3,6,3} y_{n+2,1} + s_{1,6,2} y_{n+1,1} + s_{3,7,2} y_{n,1} = 0, \\ \mathcal{G}_{11}[y_{n,1}] &\equiv s_{2,10,4} y_{n+5,1} + s_{0,10,3} y_{n+4,1} + s_{2,11,3} y_{n+3,1} + s_{0,11,2} y_{n+2,1} + s_{2,1,2} y_{n+1,1} - y_{n,1} = 0.\end{aligned}\tag{C.19}$$

D Derivation of the continuum solutions

In this subsection, we derive the continuum solutions of Theorem 8. According to Lemma 1, the monodromy free condition can be solved if the matrix $S_0^{(\text{sym})} \Gamma^{-1}$ is diagonalizable. For the continuum solutions, we solve this problem by requiring that *all the eigenvalues of the matrix $S_0^{(\text{sym})} \Gamma^{-1}$ are distinct*. This means that we require the characteristic polynomial $\mathcal{H}(x)$ of the matrix $S_0^{(\text{sym})} \Gamma^{-1}$ satisfy

$$\mathcal{H}(x) \equiv \det(xI_k - S_0^{(\text{sym})} \Gamma^{-1}) = x^k - 1. \tag{D.1}$$

The coefficients of the characteristic polynomial are related to the Stokes multipliers and here are several examples:

The 5-cut case:

$$\begin{aligned}\mathcal{H}(x) &= -1 + x^5 + x(-s_{1,3,2} + s_{0,5,2}s_{2,3,5} - s_{3,1,5}) + \\ &\quad + x^2(s_{0,3,4}s_{0,5,2} - s_{1,1,4} + s_{2,1,2}s_{2,3,5} - s_{1,3,2}s_{3,1,5} - s_{3,4,2}) + \\ &\quad + x^3(-s_{0,5,2} - s_{1,1,4}s_{1,3,2} + s_{0,3,4}s_{2,1,2} - s_{2,3,5} - s_{3,1,5}s_{3,4,2}) + \\ &\quad + x^4(-s_{0,3,4} - s_{2,1,2} - s_{1,1,4}s_{3,4,2}),\end{aligned}$$

The 7-cut case:

$$\begin{aligned}\mathcal{H}(x) &= -1 + x^7 + x(-s_{1,7,6} + s_{0,3,6}s_{2,7,3} - s_{3,4,3}) + \\ &\quad + x^2(s_{0,3,6}s_{0,7,2} - s_{1,4,2} + s_{2,4,6}s_{2,7,3} - s_{3,1,6} - s_{1,7,6}s_{3,4,3}) + \\ &\quad + x^3(-s_{1,1,5} - s_{1,4,2}s_{1,7,6} + s_{0,3,6}s_{2,1,2} + s_{0,7,2}s_{2,4,6} + s_{0,4,5}s_{2,7,3} - s_{3,1,6}s_{3,4,3} - s_{3,5,2}) + \\ &\quad + x^4(-s_{0,3,6} + s_{0,4,5}s_{0,7,2} + s_{2,1,2}s_{2,4,6} - s_{2,7,3} - s_{1,4,2}s_{3,1,6} - s_{1,1,5}s_{3,4,3} - s_{1,7,6}s_{3,5,2}) + \\ &\quad + x^5(-s_{0,7,2} - s_{1,1,5}s_{1,4,2} + s_{0,4,5}s_{2,1,2} - s_{2,4,6} - s_{3,1,6}s_{3,5,2}) + \\ &\quad + x^6(-s_{0,4,5} - s_{2,1,2} - s_{1,1,5}s_{3,5,2}).\end{aligned}\tag{D.2}$$

These equations become simpler if one uses the notation given in Eqs. (4.57) and (4.58). One can read the general formula for $k = 2m + 1$:

$$\begin{aligned}\mathcal{H}(x) &= x^k - 1 + \sum_{n=1}^m x^n \left[\sum_{i=1}^n \theta_{m+1-i}^* \tilde{\theta}_{m-n+i}^* - \sum_{i=0}^n \theta_i \tilde{\theta}_{n-i} \right] + \\ &\quad + \sum_{n=1}^m x^{k-n} \left[\sum_{i=0}^n \theta_i^* \tilde{\theta}_{n-i}^* - \sum_{i=1}^n \theta_{m+1-i} \tilde{\theta}_{m-n+i} \right],\end{aligned}\tag{D.3}$$

where we have introduced $\theta_0 \equiv 1$. Therefore, by comparing both sides of Eq. (D.1), we obtain constraints on the Stokes multipliers:

$$\begin{aligned}
0 &= \theta_m^* \tilde{\theta}_m^* - \theta_1 - \tilde{\theta}_1, \\
0 &= \theta_m^* \tilde{\theta}_{m-1}^* + \theta_{m-1}^* \tilde{\theta}_m^* - \theta_2 - \theta_1 \tilde{\theta}_1 - \tilde{\theta}_2, \\
0 &= \theta_m^* \tilde{\theta}_{m-2}^* + \theta_{m-1}^* \tilde{\theta}_{m-1}^* + \theta_{m-2}^* \tilde{\theta}_m^* - \theta_3 - \theta_2 \tilde{\theta}_1 - \theta_1 \tilde{\theta}_2 - \tilde{\theta}_3, \\
0 &= \theta_m^* \tilde{\theta}_{m-3}^* + \theta_{m-1}^* \tilde{\theta}_{m-2}^* + \theta_{m-2}^* \tilde{\theta}_{m-1}^* + \theta_{m-3}^* \tilde{\theta}_m^* - \theta_4 - \theta_3 \tilde{\theta}_1 - \theta_2 \tilde{\theta}_2 - \theta_1 \tilde{\theta}_3 - \tilde{\theta}_4, \\
&\dots
\end{aligned} \tag{D.4}$$

Since a half of the Stokes multipliers $\{\theta_n\}_{n=1}^m$ are given as

$$\theta_n = \sigma_n(\{-\omega^{n_j}\}_{j=1}^m), \tag{D.5}$$

we fix all the other Stokes multipliers $\{\tilde{\theta}_n\}_{n=1}^m$ from these constraints. Note that, since all the eigenvalues are distinct, the indices (n_1, n_2, \dots, n_m) for $\{\theta_n\}$ are also $m (= \lfloor \frac{k}{2} \rfloor)$ distinct integers. Here we can freely choose the ordering:

$$(n_1, n_2, \dots, n_m) : \quad 1 \leq n_1 < n_2 < \dots < n_m \leq k. \tag{D.6}$$

With noting the following relation:

$$\theta_m^* \theta_n = \theta_{m-n}^*, \quad m = \left\lfloor \frac{k}{2} \right\rfloor, \tag{D.7}$$

and recursively rewriting the constraint for the continuum solutions, Eqs. (D.1) and (D.3), we obtain the following simple form:

$$\tilde{\theta}_n = \mathcal{S}_n(\{\theta_j\}_{j \in \mathbb{Z}}) + \tilde{\theta}_{m-n+1}^* \theta_m^*, \quad (n = 1, 2, \dots, m), \tag{D.8}$$

with the polynomials $\mathcal{S}_n(x)$ (defined by Eq. (4.81)). This results in Theorem 8.

E Calculation in the 3-cut $(1, 1)$ critical point ($r = 2$)

The specialty of the 3-cut $(1, 1)$ critical point is that the symmetric Stokes sectors D_{4n} (see Eq. (3.15)) do not cover the whole plane \mathbb{C} . Therefore, we consider doubling of the sectors

$$D_{2n}, \quad S_{2n}^{(\text{sym})} \equiv S_{2n} S_{2n+1}, \quad (n = 0, 1, \dots, 5), \tag{E.1}$$

and express the boundary condition (4.33) as follows:

$$\begin{aligned}
Y^{(4n)} &= \begin{pmatrix} y_{4n,1} \\ y_{4n,2} \\ y_{4n,3} \end{pmatrix} \equiv \Gamma^n X^{(4n)} = \begin{pmatrix} x_{n+1}^{(4n)} \neq 0 \\ x_{n+2}^{(4n)} \\ x_{n+3}^{(4n)} = 0 \end{pmatrix}, \\
Y^{(4n+2)} &= \begin{pmatrix} y_{4n+2,1} \\ y_{4n+2,2} \\ y_{4n+2,3} \end{pmatrix} \equiv \Gamma^n X^{(4n+2)} = \begin{pmatrix} x_{n+1}^{(4n+2)} \\ x_{n+2}^{(4n+2)} \neq 0 \\ x_{n+3}^{(4n+2)} = 0 \end{pmatrix},
\end{aligned} \tag{E.2}$$

with

$$X^{(2n)} = S_{2n}^{(\text{sym})} X^{(2n+2)}, \quad (n = 0, 1, \dots, 5). \quad (\text{E.3})$$

This is then written as

$$\begin{aligned} Y^{(4n)} &= S_0^{(\text{sym})} Y^{(4n+2)}, & Y^{(4n+2)} &= (S_2^{(\text{sym})} \Gamma^{-1}) Y^{(4n+4)}, \\ \Leftrightarrow \quad \begin{cases} y_{4n,i} &= y_{4n+2,i} + \sum_j \left[s_{0,i,j}^{(\text{sym})} \times y_{4n+2,j} \right], \\ y_{4n+2,i} &= y_{4n+4,i-1} + \sum_j \left[s_{2,i,j}^{(\text{sym})} \times y_{4n+4,j-1} \right]. \end{cases} \end{aligned} \quad (\text{E.4})$$

These recursion relations are expressed as

$$\begin{aligned} y_{4n,3} &= y_{4n+2,3} = 0, & y_{4n,1} &= y_{4n+2,1} \neq 0, \\ y_{4n,2} &= y_{4n+2,2} \neq 0, & y_{4n+2,2} &= y_{4n+4,1} \neq 0, \end{aligned} \quad (\text{E.5})$$

and the following two recursion equation for $y_{4n,1}$

$$y_{4n,1} = s_{2,1,2} \times y_{4n+4,1}, \quad y_{4n+4,1} = -s_{3,3,2} \times y_{4n,1}. \quad (\text{E.6})$$

As one may notice, this equation itself is the same as Eq. (4.37). The solutions (labeled by l) to this boundary condition is easily solved as

$$y_{4n,1}^{(l)} = \omega^{nl}, \quad s_{3,3,2}^{(l)} = -\omega^l, \quad s_{2,1,2}^{(l)} = \omega^{-l}, \quad (l = 0, 1, 2), \quad (\text{E.7})$$

and the general solution is given as

$$s_{0,2,3}^{(l)} = -\omega^{-l} + \omega^l (s_{1,1,3}^{(l)})^*, \quad (\text{E.8})$$

with Eq. (E.7). This provides the first case of the continuum solution (D.8).

F Calculation in the 4-cut $(1, 1)$ critical point ($r = 2$)

Here we calculate the 4-cut $(1, 1)$ critical point as an example in which the coprime condition of Eq. (3.8) is violated:

$$(k, r) = (4, 2). \quad (\text{F.1})$$

In this case, the leading exponents are degenerate:

$$\varphi^{(1)}(t; \zeta) \sim \varphi^{(3)}(t; \zeta), \quad \varphi^{(2)}(t; \zeta) \sim \varphi^{(4)}(t; \zeta), \quad (\text{F.2})$$

and we consider the subleading Stokes lines:

$$\text{Re} \left[(\varphi_{-r+1}^{(1)} - \varphi_{-r+1}^{(3)}) \zeta^{r-1} \right] = 0, \quad \text{Re} \left[(\varphi_{-r+1}^{(2)} - \varphi_{-r+1}^{(4)}) \zeta^{r-1} \right] = 0. \quad (\text{F.3})$$

The dominance profile in the ζ plane is shown in Fig. 12.

Here we use the fine Stokes sectors D_n (calculated in the leading Stokes lines) which are defined as

$$D_n \equiv D \left(\frac{(n-1)\pi}{4}, \frac{n\pi}{4} \right), \quad n = 0, 1, 2, 3. \quad (\text{F.4})$$

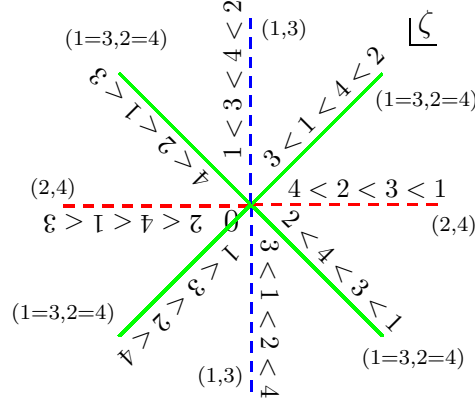


Figure 12: The dominance profile in the 4-cut (1,1) case in terms of ζ . The bold lines express the leading Stokes lines with degeneracy $\varphi^{(1)} \sim \varphi^{(3)}$ and $\varphi^{(2)} \sim \varphi^{(4)}$. The dashed lines express the sub leading Stokes lines for (1,3) and (2,4).

All fine Stokes matrices can be expressed in terms of S_0 as

$$S_n = \Gamma^{-n} S_0 \Gamma^n, \quad S_0 = \begin{pmatrix} 1 & & & \\ \alpha & 1 & \beta & \\ \epsilon & & 1 & \\ \gamma & & \delta & 1 \end{pmatrix}. \quad (\text{F.5})$$

Then the multi-cut boundary condition is given as

$$Y^{(n)} \equiv \Gamma^n X^{(n)} = \begin{pmatrix} y_{n,1} \neq 0 \\ y_{n,2} = 0 \\ y_{n,3} = 0 \\ y_{n,4} \neq 0 \end{pmatrix}. \quad (\text{F.6})$$

The recursive equations are expressed as

$$y_{n,1} = y_{n+1,4}, \quad 0 = \epsilon \times y_{n+1,4}, \quad y_{n+1,1} + \alpha \times y_{n,1} = 0, \quad \gamma \times y_{n+1,1} - y_{n,1} = 0 \quad (\text{F.7})$$

and there are four solutions which are labeled by l ($\alpha \rightarrow \alpha^{(l)}$):

$$\alpha^{(l)} = -\omega^l, \quad \gamma^{(l)} = \omega^{-l}, \quad \epsilon^{(l)} = 0, \quad y_{n,1}^{(l)} = \omega^{nl} \quad (l = 0, 1, 2, 3). \quad (\text{F.8})$$

By directly solving the monodromy free condition, the other Stokes multipliers ($\beta^{(l)}$ and $\gamma^{(l)}$) are also fixed and the solution is given as

$$S_0 = \begin{pmatrix} 1 & & & \\ -\omega^l & 1 & -\omega^{-l} & \\ 0 & & 1 & \\ \omega^{-l} & & \omega^l & 1 \end{pmatrix}, \quad (l = 0, 1, 2, 3). \quad (\text{F.9})$$

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